

Unit 4

Conics

Introduction

This unit is about three types of curve – the *parabola*, the *ellipse* and the *hyperbola*. (Each of these names is pronounced with the emphasis on the second syllable.) An example of each type of curve is shown in Figure 1. The third curve, a hyperbola, has two parts.

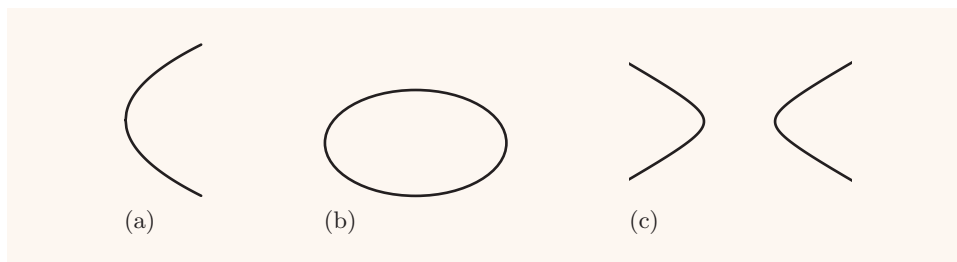


Figure 1 (a) A parabola (b) an ellipse (c) a hyperbola

These types of curves have been known for over two thousand years, since the time of the Ancient Greeks. It's useful to study them because they appear in a wide variety of situations. For example, parabolas occur as the paths of objects that are thrown and as the paths of droplets of water in a fountain. Ellipses occur as the shapes of the orbits of planets and comets, and hyperbolas occur as the paths of subatomic particles. You can see conics in your home too, such as in the shapes that the light from a lamp throws on a wall.

These curves are called *conic sections*, or *conics* for short, since they can all be obtained by slicing a *double cone* in different ways. In Figure 2 a double cone is sliced by a plane to give a cross-section that's an ellipse.

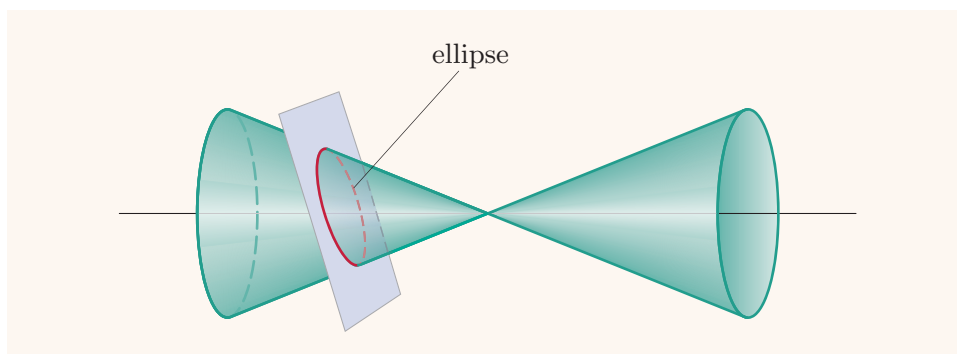


Figure 2 A plane slicing a double cone and forming an ellipse

You'll see in Section 1 that by slicing a double cone in other ways you can obtain cross-sections that are parabolas or hyperbolas.



Parabolas in a fountain



Curves from a lamp

You saw in MST124 Unit 5 that circles can be described both geometrically and algebraically. Geometrically, the circle with centre (a, b) and radius r is the set of points that are at distance r from the point (a, b) . Algebraically, the same circle is the set of points whose coordinates (x, y) satisfy the equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

In Sections 2, 3 and 4 of this unit you'll learn how parabolas, ellipses and hyperbolas can be described both geometrically and algebraically. You'll also see that the geometric properties of conics lead to many practical applications, such as in the design of solar ovens, telescopes and medical equipment.

We'll mainly consider conics positioned in the plane in the way described below, as illustrated in Figure 3.

- For a parabola: its axis of symmetry is the x -axis, it passes through the origin and its other points lie to the right of the origin.
- For an ellipse: its axes of symmetry are the x - and y -axes, and its largest width is along the x -axis;
- For a hyperbola: its axes of symmetry are the x - and y -axes, and it crosses the x -axis.

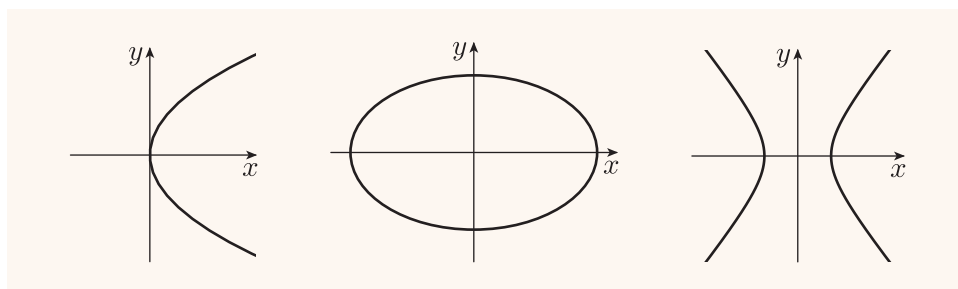


Figure 3 Conics in standard position

Conics in these positions are said to be in *standard position*. Any conic, lying anywhere in the plane, can be rotated and translated so that it is in standard position. As you'll see, the equations of conics in standard position have straightforward and easily recognisable forms. The equations of conics in other positions are more complicated, and beyond the scope of this module, though they're mentioned briefly in Section 5.

In Section 6, you'll learn about an alternative algebraic way of describing conics, and other curves and straight lines. You'll see how, instead of describing a curve using a single equation that expresses the relationship between the x - and y -coordinates of each point on the curve, you can use *two* equations. One equation expresses the x -coordinate of each point on the curve in terms of a third variable, often denoted by t , and the other equation expresses the corresponding y -coordinate in terms of the same third variable. This powerful technique is known as *parametrisation*. As well as being used to describe conics, it can also be used to describe much more intricate curves than those that you've met so far.

1 Conic sections

In this short first section you'll look at the different types of cross-section that you can obtain by slicing a double cone with a plane.

A **double cone** consists of two identical, infinitely long, hollow circular cones joined at their vertices and aligned symmetrically along a straight line, as shown in Figure 4. The point at which the two cones meet is called the **apex** or **vertex** of the double cone. The straight line that passes centrally through the double cone is called its **axis**.

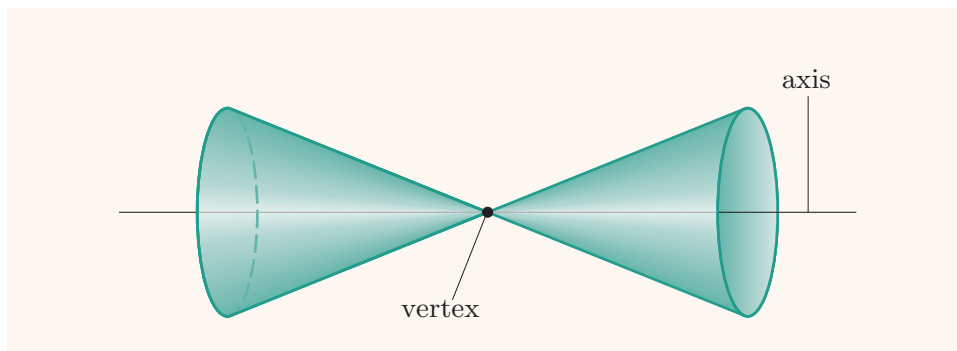


Figure 4 A double cone

The 'width' of a double cone is determined by its **opening angle** (sometimes called its **aperture**), which is the angle at the vertex obtained when the cone is sliced by a plane that passes through its axis, as shown in Figure 5.

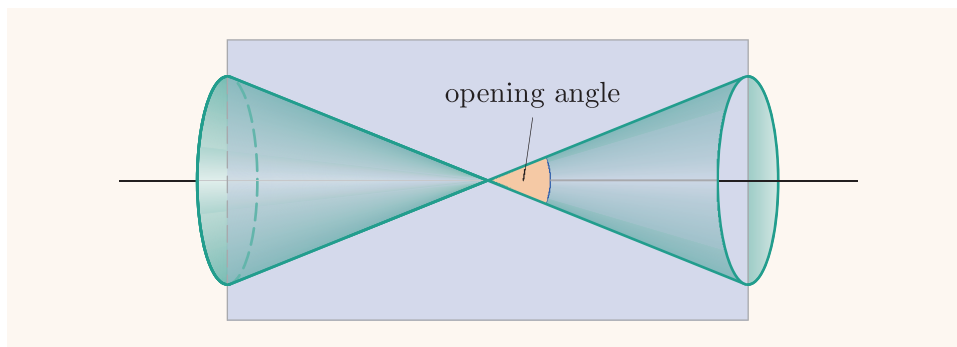


Figure 5 The opening angle of a cone

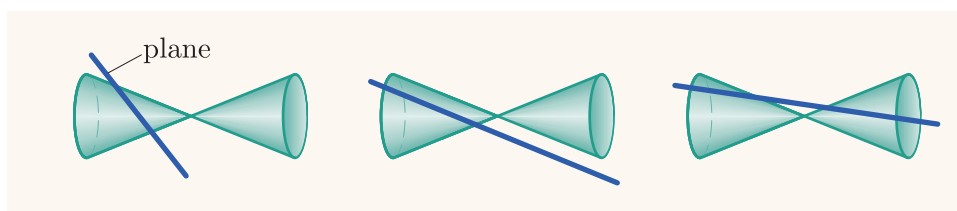
In the next activity you can investigate the different cross-sections that you can obtain by slicing a double cone with a plane.



Activity 1 *Slicing a double cone*

Open the *Slicing a double cone* applet. It shows a double cone sliced by a plane, with the resulting cross-section displayed underneath. You can move the plane to change the cross-section, rotate the double cone for a different view, and change the opening angle of the double cone.

- (a) Position the plane at various angles in turn, such as those shown in the diagrams below, but always so that the plane does *not* pass through the apex of the double cone. In each case observe the cross-section that you obtain. The diagrams show the plane tilted at an angle greater than, equal to, and less than that of the side of the double cone, respectively.



- (b) Now move the plane so that it passes through the apex of the double cone, and experiment by tilting the plane at different angles, as in part (a), to see what types of cross-section you can obtain.

When you have tried this activity, read the descriptions in the text below of what you should have observed.

In Activity 1 you should have found that by positioning the slicing plane in various ways, you can obtain the following types of curves. These curves are called **conic sections**, or **conics** for short.

Circle

If you position the slicing plane so that it is perpendicular to the axis of the double cone, but does not pass through the apex, then the cross-section is a circle, as shown in Figure 6.

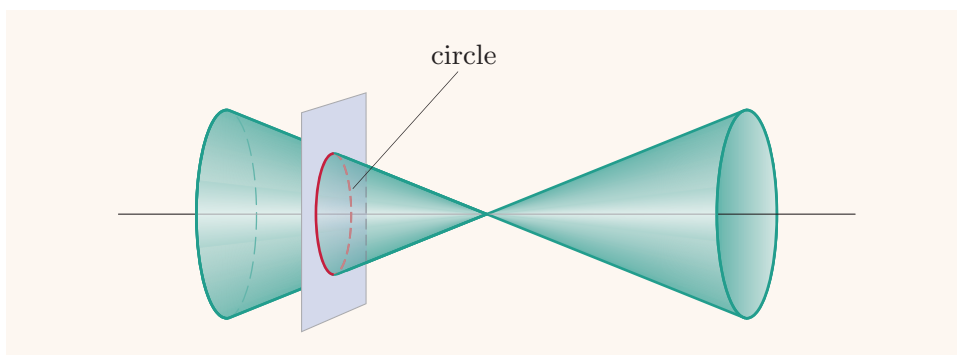


Figure 6 A circle as a cross-section

Ellipse

If you then tilt the plane slightly (by rotating it about a horizontal straight line that it passes through, but making sure that the plane doesn't pass through the apex of the double cone), then you obtain a curve like the one shown in Figure 7. This curve is called an **ellipse**. The more you tilt the plane, the more elongated is the ellipse.

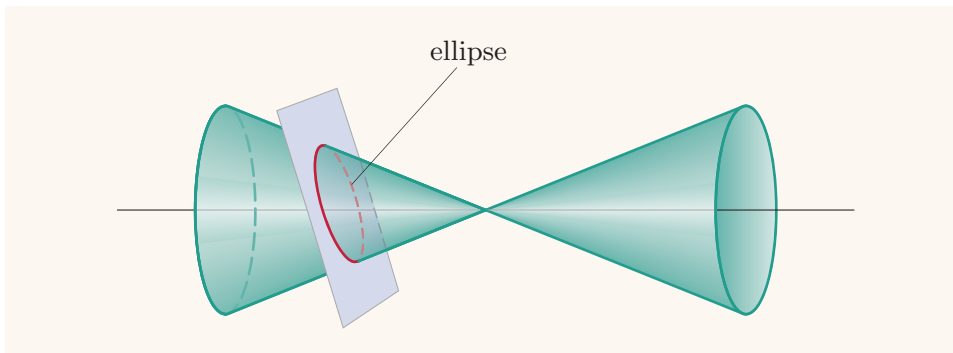


Figure 7 An ellipse as a cross-section

Parabola

If you continue to tilt the plane, again ensuring that it doesn't pass through the apex, then you reach a situation in which it is tilted at the same angle as the side of the double cone, as shown in Figure 8. In this case, you obtain a curve called a **parabola**, which extends indefinitely.

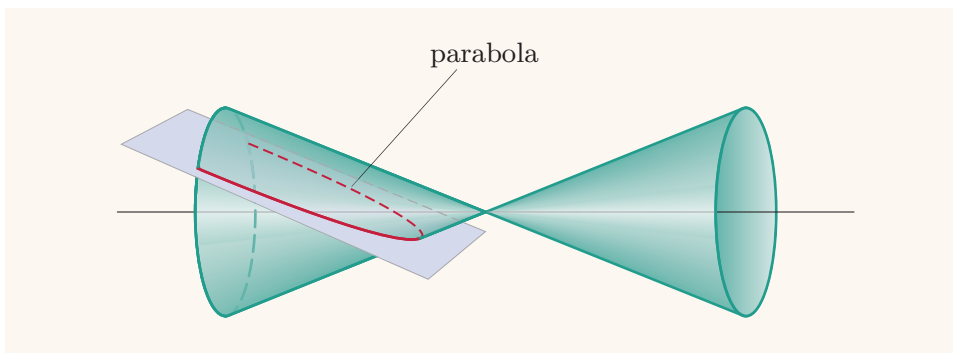


Figure 8 A parabola as a cross-section

Hyperbola

If you now tilt the plane further, at any angle up to and including the plane being horizontal, but not passing through the apex, then the cross-section is a curve that has two parts, one on each side of the apex, as shown in Figure 9. This curve is called a **hyperbola**. Each part extends indefinitely.

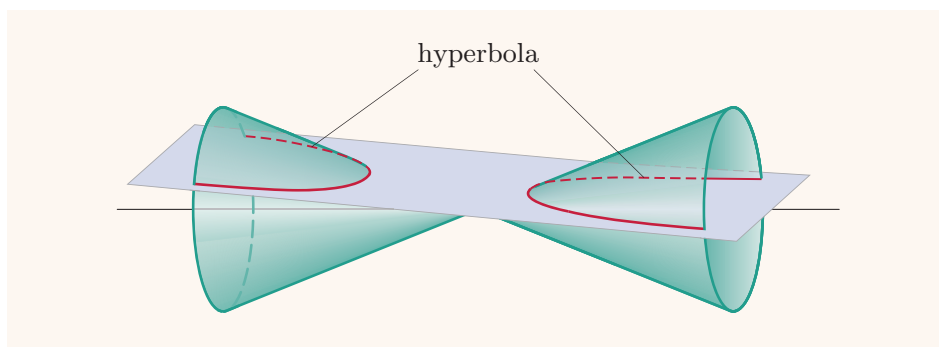


Figure 9 A hyperbola as a cross-section

In general, however you rotate and move the plane, as long as you don't make it pass through the apex of the double cone, the cross-section is a circle, an ellipse, a parabola or a hyperbola.

Degenerate conics

If you now move the slicing plane so that it passes through the apex of the double cone, then the cross-section that you obtain is one of the three shapes shown at the bottom of Figure 10. If you start with the plane vertical, or tilted slightly from the vertical, but not enough to meet the side of the cone, then the cross-section is a single point, as shown in Figure 10(a). If you tilt the plane further, just enough to lie along the side of the cone, then the cross-section is a straight line, as shown in Figure 10(b). If you tilt the plane even further, then the cross-section consists of two straight lines intersecting at the apex, as shown in Figure 10(c). Cross-sections of these three types are called **degenerate conic sections** or **degenerate conics**. We won't consider them further in this unit.

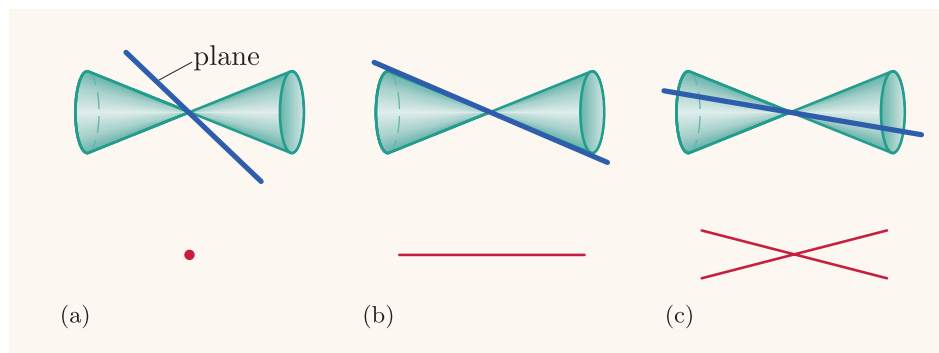
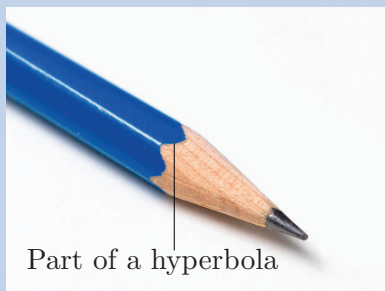


Figure 10 Degenerate conics as cross-sections

A conic that is not a degenerate conic is called a **non-degenerate conic**.

Conic sections can appear in some surprising places! The photograph below shows a pencil with flat sides, which has been sharpened in the usual way to make it conical at one end. On each flat side of the pencil, the boundary between the painted side and the conical end is formed by a plane slicing a cone. The plane is parallel to the axis of the cone, so the boundary curve is part of a hyperbola.



In the next three sections, you'll consider the main geometric and algebraic properties of parabolas, ellipses and hyperbolas in turn. Each section starts with a geometric definition for one of these three types of conics. This definition is not the definition involving a double cone that you've just met, but a different, though equivalent, geometric definition. These alternative definitions for conics are known as their *focus-directrix* definitions, for reasons that you'll see. It's quite complicated to prove that the two sets of definitions are equivalent, and the details are not included in this unit.

2 Parabolas

You know quite a lot about parabolas already, perhaps from MST124 Unit 2. You saw there that the graph of an equation of the form

$$y = ax^2 + bx + c,$$

where a , b and c are constants with $a \neq 0$, is a parabola whose axis of symmetry is vertical and whose y -intercept is c . This is illustrated in Figure 11 for the two possible cases $a > 0$ and $a < 0$, which give 'u-shaped' and 'n-shaped' parabolas, respectively. Remember that an **axis of symmetry** of a flat shape is a straight line such that if you reflect the shape in this line then the shape looks unchanged. Remember also that a **y -intercept** of a curve is a value of y at which it intersects the y -axis, and similarly an **x -intercept** of a curve is a value of x at which it intersects the x -axis.

The **vertex** of a parabola is the point where it intersects its axis of symmetry.

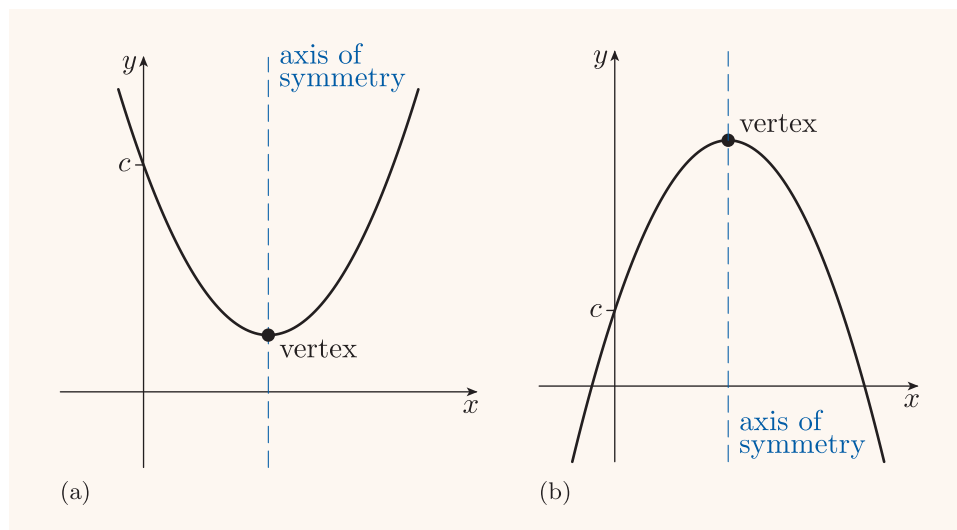


Figure 11 Graphs of equations of the form $y = ax^2 + bx + c$ with (a) $a > 0$ (b) $a < 0$

A parabola doesn't have to have a vertical axis of symmetry. In this section you'll see how the shape known as a parabola can be defined more generally. You'll also discover some of the properties of parabolas and their practical uses.

2.1 The focus–directrix definition of a parabola

You've seen that a circle can be defined geometrically as the set of points that are a fixed distance from a particular point, called the centre of the circle, as illustrated in Figure 12.

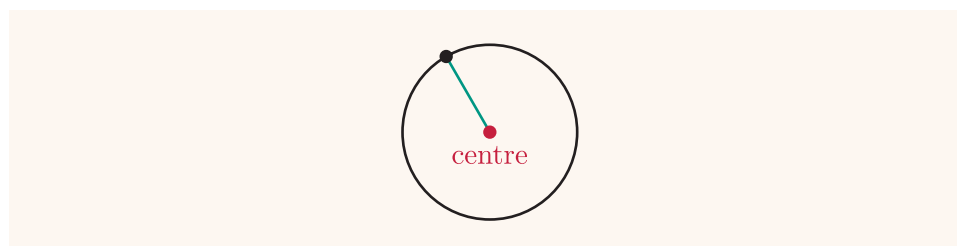


Figure 12 A circle

Now suppose that instead of starting with a point, you start with a point and also a line that doesn't pass through the point. The shape formed by the set of points that are the same distance from the point as they are from the line, as illustrated in Figure 13, is called a **parabola**. The point that you started with is called the **focus** of the parabola, and the line that you started with is called the **directrix** of the parabola.

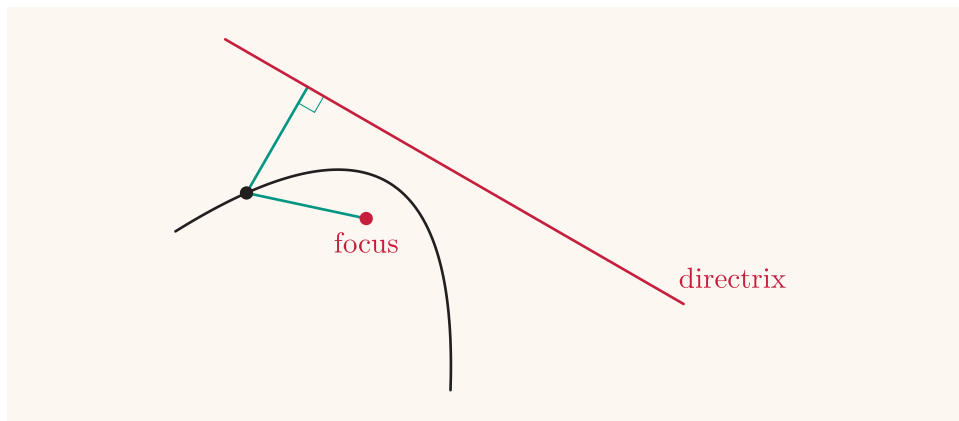


Figure 13 A parabola obtained from a focus and directrix

Note that the distance of a point from a line is defined to be the *shortest* distance from the point to the line. This distance is the length of the perpendicular line segment from the point to the line, as illustrated in Figure 13.

Figure 14 illustrates some notation that we'll use throughout this unit. We denote the distance from a point P to a line d by Pd , and we denote the distance from a point P to another point F by PF .

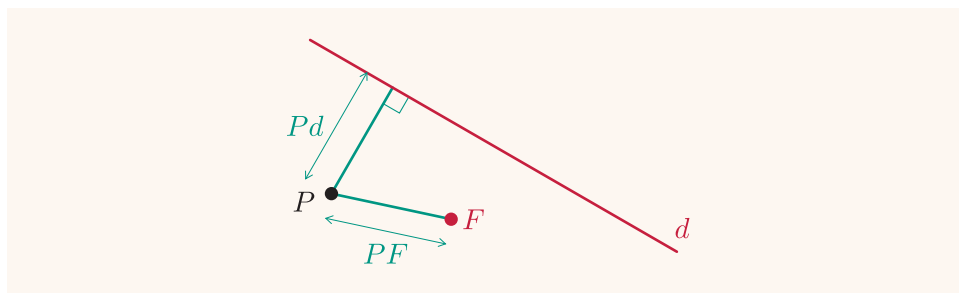


Figure 14 Notation for distances

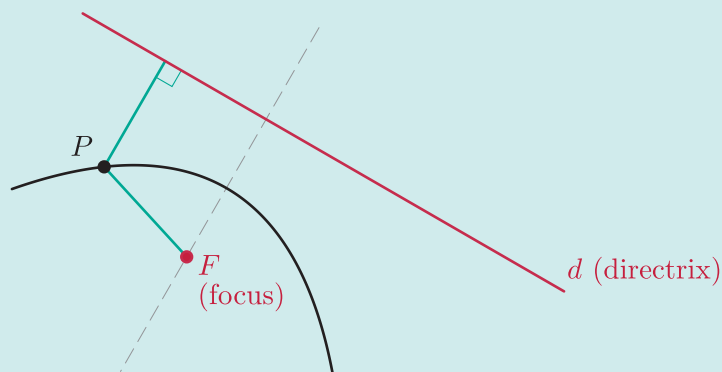
The focus–directrix definition of a parabola is summarised in the following box.

The focus–directrix definition of a parabola

Suppose that F is a point and d is a line that does not pass through F . Then the curve formed by the points satisfying the equation

$$PF = Pd$$

is a **parabola** with **focus** F and **directrix** d .



You can explore this definition in the next activity.



Activity 2 Exploring the focus–directrix definition of a parabola

Open the *Exploring the focus–directrix definition of a parabola* applet.

It shows a point F (the focus), a line d (the directrix) and the curve formed by the points that satisfy the equation $PF = Pd$. You can opt for the applet to show a particular point P on the curve, and its distances from F and d . You can move the focus F and the point P , and you can move and rotate the directrix d .

- Check that the curve looks as you would expect for a parabola. Drag the point P along the curve, and check that its distances from F and d always seem to be equal.
- Experiment with moving the focus and directrix to see some different parabolas that you can obtain.

You can use the focus–directrix definition of a parabola to find the equation of a particular parabola from the coordinates of its focus and the equation of its directrix, as illustrated in the next example. Remember that the notation $P(x, y)$ denotes the point P with coordinates (x, y) . Remember also that the distance between any two points in the plane is given by the *distance formula*, as follows.

Distance formula

The distance between the points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 1 *Finding the equation of a parabola*

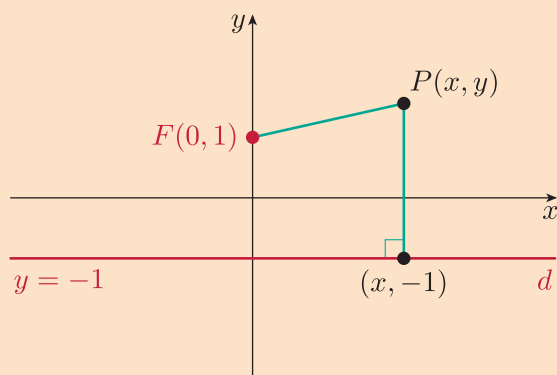
Find the equation of the parabola with focus $(0, 1)$ and directrix $y = -1$.

Solution

Let F be the focus, $(0, 1)$, and let d be the directrix, $y = -1$.

Let $P(x, y)$ be any point in the plane.

 Use the distance formula to find expressions for the distances PF and Pd . 



The distance of P from F is

$$PF = \sqrt{(x - 0)^2 + (y - 1)^2} = \sqrt{x^2 + (y - 1)^2}.$$

The distance of P from d is its distance from the point $(x, -1)$, which is

$$Pd = \sqrt{(x - x)^2 + (y - (-1))^2} = \sqrt{(y + 1)^2}.$$

 Use these expressions to find the equation of the parabola. 

The point $P(x, y)$ lies on the parabola whenever it satisfies the equation

$$PF = Pd;$$

that is,

$$\sqrt{x^2 + (y - 1)^2} = \sqrt{(y + 1)^2}.$$

Simplifying gives

$$x^2 + (y - 1)^2 = (y + 1)^2$$

$$x^2 + y^2 - 2y + 1 = y^2 + 2y + 1$$

$$x^2 = 4y$$

$$y = \frac{1}{4}x^2.$$

Hence the equation of the parabola is $y = \frac{1}{4}x^2$.

Note that the expression $\sqrt{(y + 1)^2}$ that was found for the distance of a general point $P(x, y)$ from the directrix $y = -1$ in Example 1 could have been simplified to the expression $|y + 1|$, which means the magnitude of $y + 1$. It can't be simplified to the expression $y + 1$, because the value of $y + 1$ might be negative. However, because of the purpose for which this expression is used in Example 1, it's just as convenient to leave it in the form $\sqrt{(y + 1)^2}$, as was done in Example 1. There are several similar instances throughout this unit.

Note also that removing the square root sign from each side of the equation in Example 1 (that is, squaring both sides of the equation) does give an equivalent equation, because the values of the expressions under the square root signs are always non-negative. Again, there are several similar instances throughout this unit.

The equation found in Example 1 is of the form $y = ax^2 + bx + c$, with $a = \frac{1}{4}$, $b = 0$ and $c = 0$, so it's the equation of a parabola with a vertical axis of symmetry and its vertex at the origin O . This parabola is shown in Figure 15, with its focus and directrix.

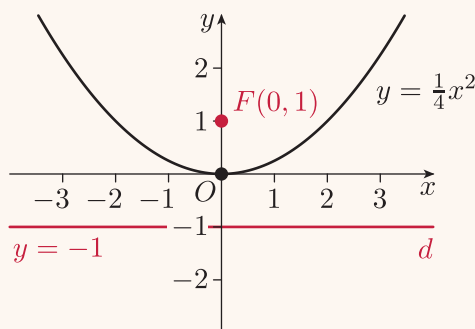


Figure 15 The parabola with equation $y = \frac{1}{4}x^2$

The directrix of the parabola in Example 1 is horizontal. In the next activity you can try finding the equation of a parabola with a vertical directrix.

Activity 3 Finding the equation of a parabola

Find the equation of the parabola with focus $(5, 0)$ and directrix $x = -5$.

2.2 Parabolas in standard position

In Example 1 and Activity 3 in the previous subsection, the directrix of the parabola was parallel to one of the coordinate axes, which made it relatively straightforward to find an expression for the distance of a general point $P(x, y)$ from the directrix, and hence to find the equation of the parabola. If the directrix is slant, however, then it's more tricky to do this.

For this reason, we'll now concentrate on parabolas that are positioned in the plane in a certain way that makes it straightforward to find and work with their equations.

Given any parabola, positioned anywhere in the plane, as illustrated in Figure 16(a), you can first rotate it, along with its focus and directrix, until the directrix is vertical, with the focus to the right of the directrix, as illustrated in Figure 16(b). Then you can translate it vertically until the focus lies on the x -axis, as illustrated in Figure 16(c). Finally, you can translate it horizontally until the focus and directrix are at equal distances on either side of the origin, as illustrated in Figure 16(d). The resulting parabola is said to be in **standard position**. Every parabola is congruent to (that is, is the same size and shape as) a parabola in standard position.

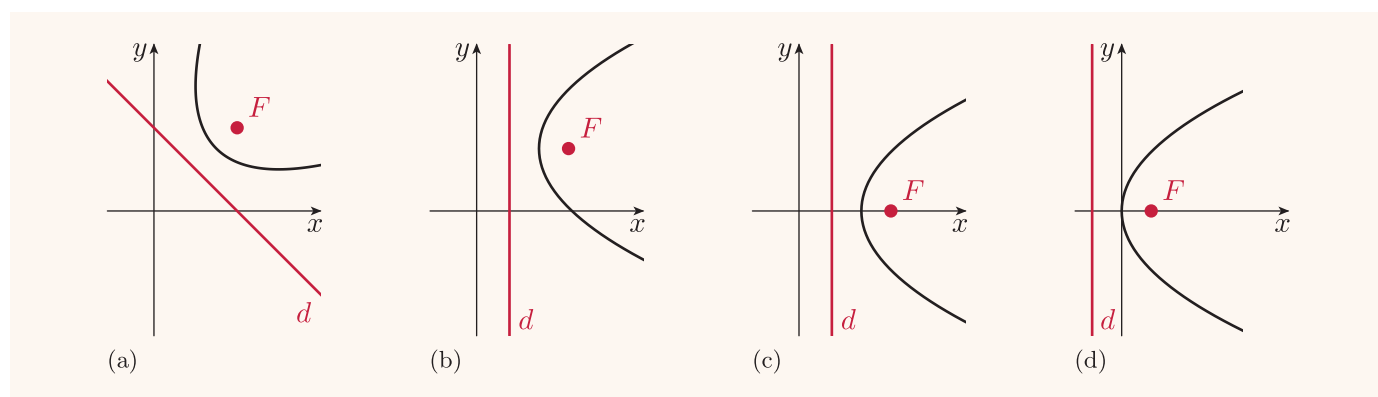


Figure 16 A parabola rotated and translated into standard position

It's straightforward to find the equation of any parabola in standard position by using the method of Example 1 and Activity 3. However, instead of using this method to individually find the equation of each parabola in standard position that we want to work with, it's more convenient to use the same method to find a *general* equation for a

parabola in standard position, in terms of the x -coordinate, say a , of its focus. Then we can quickly find the equation of any particular parabola in standard position, by substituting in the appropriate value of a . This is a useful type of approach that we often take in mathematics.

So let's now find such a general equation. Consider a parabola in standard position, with focus F and directrix d . Suppose that F has x -coordinate a . In other words, F is the point $(a, 0)$, where $a > 0$, as illustrated in Figure 17. Then, since the focus and directrix are at equal distances on either side of the origin, the directrix d has the equation $x = -a$, as also shown in Figure 17.

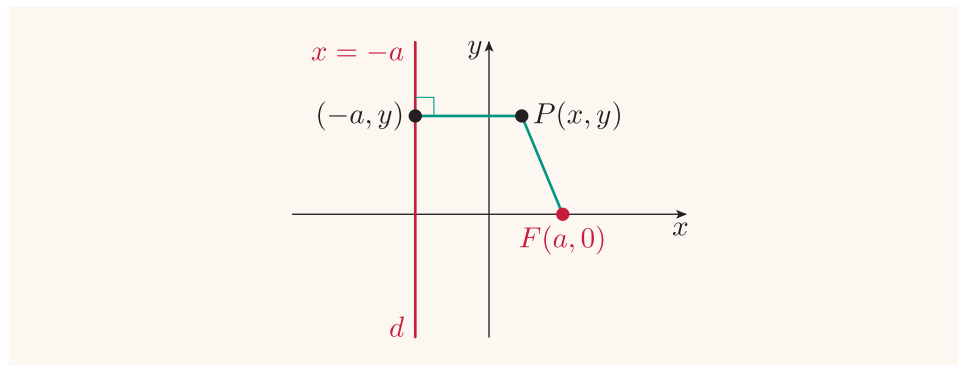


Figure 17 The focus $(a, 0)$, the directrix $x = -a$ and a point $P(x, y)$

We can now use the method of Example 1 to find the equation of the parabola.

Let $P(x, y)$ be any point in the plane, as shown in Figure 17.

The distance of P from F is

$$PF = \sqrt{(x - a)^2 + (y - 0)^2} = \sqrt{(x - a)^2 + y^2}.$$

The distance of P from d is its distance from the point $(-a, y)$, which is

$$Pd = \sqrt{(x - (-a))^2 + (y - y)^2} = \sqrt{(x + a)^2}.$$

The point $P(x, y)$ lies on the parabola whenever it satisfies the equation

$$PF = Pd;$$

that is,

$$\sqrt{(x - a)^2 + y^2} = \sqrt{(x + a)^2}.$$

Simplifying gives

$$\begin{aligned} (x - a)^2 + y^2 &= (x + a)^2 \\ x^2 - 2ax + a^2 + y^2 &= x^2 + 2ax + a^2 \\ y^2 &= 4ax. \end{aligned}$$

So the equation of the parabola is $y^2 = 4ax$.

This is the standard equation for a parabola in standard position.

The equation of a parabola in standard position

The parabola in standard position with focus $(a, 0)$ (where $a > 0$) and directrix $x = -a$ has equation

$$y^2 = 4ax.$$

For example, the parabola in standard position whose focus is $(\frac{3}{2}, 0)$, and whose directrix is therefore the line $x = -\frac{3}{2}$, has the equation

$$y^2 = 4 \times \frac{3}{2}x; \quad \text{that is,} \quad y^2 = 6x.$$

This parabola is illustrated in Figure 18.

Activity 4 Writing down equations of parabolas in standard position

In each of parts (a) and (b), write down the equation of the parabola in standard position with the given focus or directrix.

- (a) Focus $(\frac{1}{4}, 0)$ (b) Directrix $x = -2$

The equation $y^2 = 6x$ of the parabola in Figure 18 is an equation in **implicit form**, because it doesn't express either of the two variables x and y explicitly in terms of the other variable. If you want, you can rearrange it into the equation $x = \frac{1}{6}y^2$, which expresses x as a function of y . This equation is in **explicit form**, because it expresses one of its two variables as a function of the other. However, you can't rearrange the equation $y^2 = 6x$ to express y as a function of x , because this equation gives two values of y for every value of x .

In general, every equation of the form $y^2 = 4ax$, for a parabola in standard position, is in implicit form (unless $a = \frac{1}{4}$). You can rearrange it into an equation that expresses x as a function of y , but you can't rearrange it into an equation that expresses y as a function of x .

2.3 Basic properties of parabolas

You can determine many properties of a parabola in standard position by using its equation $y^2 = 4ax$ (where $a > 0$).

First, note that no point (x, y) where x is negative satisfies the equation $y^2 = 4ax$, because y^2 is never negative. So no part of the parabola lies to the left of the y -axis. Also, you can check that the points $(a, 2a)$ and $(a, -2a)$ satisfy the equation $y^2 = 4ax$ and therefore lie on the parabola. This fact can be helpful when you want to sketch a parabola in standard position from its equation.

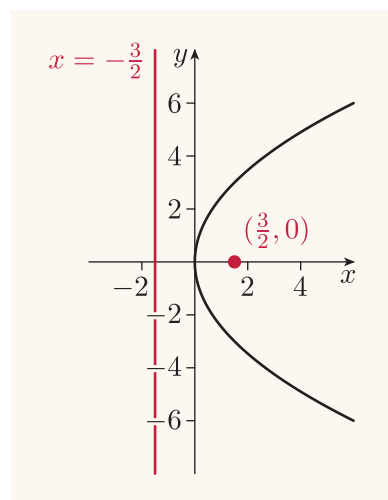


Figure 18 The parabola with equation $y^2 = 6x$

To obtain further useful facts about the shape of a parabola, it's helpful to rearrange the equation $y^2 = 4ax$ of a parabola in standard position.

Although, as you've seen, you can't write it as an equation that expresses y as a function of x , you can rewrite it as *two separate* equations, each expressing y as a function of x , as follows:

$$y = 2\sqrt{ax} \quad \text{and} \quad y = -2\sqrt{ax}.$$

The first equation here is the equation of the top half of the parabola (where $y \geq 0$), and the second equation is the equation of the bottom half (where $y \leq 0$). The two equations can be written together as $y = \pm 2\sqrt{ax}$.

These two equations show that the bottom half of the parabola is the reflection of the top half in the x -axis; that is, the x -axis is an axis of symmetry of the parabola. It follows that the vertex of the parabola is the origin O . Also, since the value of the expression $2\sqrt{ax}$ increases as x increases, the top half of the parabola slopes up indefinitely, and hence the bottom half slopes down indefinitely.

Note that the parabola is **unbounded**. This means that it's not possible to enclose it within a circle, no matter how large you take the circle to be. Here's a summary of the properties of a parabola in standard position that you've met.

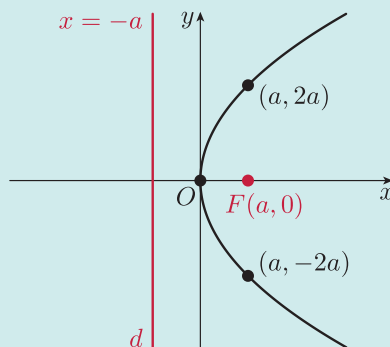
Properties of a parabola in standard position

The parabola with equation

$$y^2 = 4ax \quad (\text{where } a > 0),$$

has the following properties:

- its vertex is the origin
- its axis of symmetry is the x -axis
- it passes through the points $(a, \pm 2a)$
- it consists of one unbounded part
- its focus is $(a, 0)$
- its directrix is $x = -a$.



Note that these properties of a parabola in standard position also tell you about the properties of *any* parabola (since every parabola is congruent to a parabola in standard position). For example, they tell you that every parabola has an axis of symmetry.

If you have an equation that you can rearrange into the form $y^2 = 4ax$, then you can recognise it as the equation of a parabola in standard position, and use the properties above to help you sketch this parabola, as illustrated in the next example.

When you sketch a parabola in standard position, you should draw it smoothly, and symmetrically above and below the x -axis. You should mark and label its vertex O , either with the letter O or with its coordinates $(0, 0)$, and you should mark and label the points $(a, 2a)$ and $(a, -2a)$, with their particular coordinates. You should label the parabola with its equation, or include the equation in a caption. If you include the focus and directrix in your sketch, then you should label the focus with its coordinates and the directrix with its equation. All of this is illustrated in the next example.

Example 2 *Sketching a parabola in standard position*

Consider the equation

$$3y^2 - 4x = 0.$$

- Show that this equation represents a parabola in standard position.
- Sketch the parabola.
- Write down the coordinates of the focus and the equation of the directrix of the parabola, and add these features to your sketch.

Solution

-  Rearrange the given equation into the form $y^2 = 4ax$, and identify the value of a . 

Rearranging the equation gives

$$y^2 = \frac{4}{3}x;$$

that is,

$$y^2 = 4 \times \frac{1}{3}x.$$

This equation is of the form $y^2 = 4ax$ with $a = \frac{1}{3}$. Hence it is the equation of a parabola in standard position.

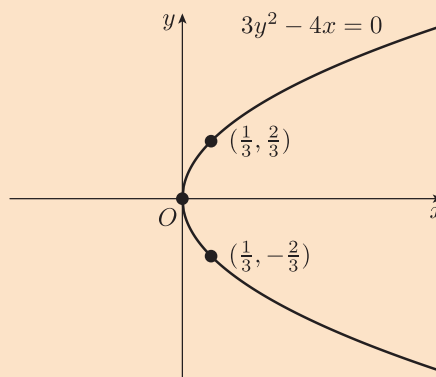


- (b) To help you sketch the parabola, first write down the vertex $(0, 0)$, and the points $(a, \pm 2a)$ through which the parabola passes.

The vertex is $(0, 0)$. The parabola passes through the points $(\frac{1}{3}, \pm \frac{2}{3})$.

Mark and label these points, and hence sketch the parabola.

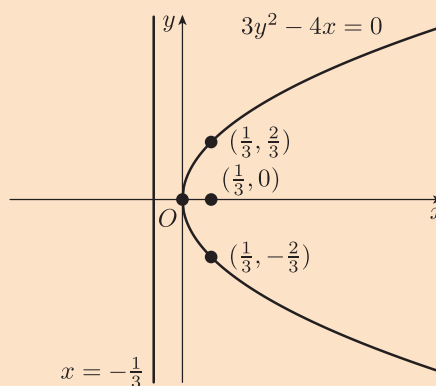
A sketch of the parabola is shown below.



- (c) The focus is $(a, 0)$ and the directrix is $x = -a$.

The focus is $(\frac{1}{3}, 0)$ and the directrix is $x = -\frac{1}{3}$.

Draw and label the focus and directrix.



You can practise sketching parabolas in standard position in the next two activities.

Activity 5 *Sketching a parabola in standard position*

Consider the equation

$$y^2 = x.$$

- Show that this equation represents a parabola in standard position.
- Sketch the parabola.
- Write down the coordinates of the focus and the equation of the directrix, and add these features to your sketch.

Activity 6 *Sketching another parabola in standard position*

Repeat Activity 5 for the equation

$$2y^2 - x = 0.$$

You've seen that an equation of the form $y^2 = 4ax$ where a is positive is the equation of a parabola in standard position. An equation of the form $y^2 = 4ax$ where a is *negative* is *not* the equation of a parabola in standard position, but it's still the equation of a parabola. This parabola is the reflection in the y -axis of a parabola in standard position. For example, the equation $y^2 = -6x$ represents the parabola that is the reflection in the y -axis of the parabola in standard position with equation $y^2 = 6x$, as illustrated in Figure 19.

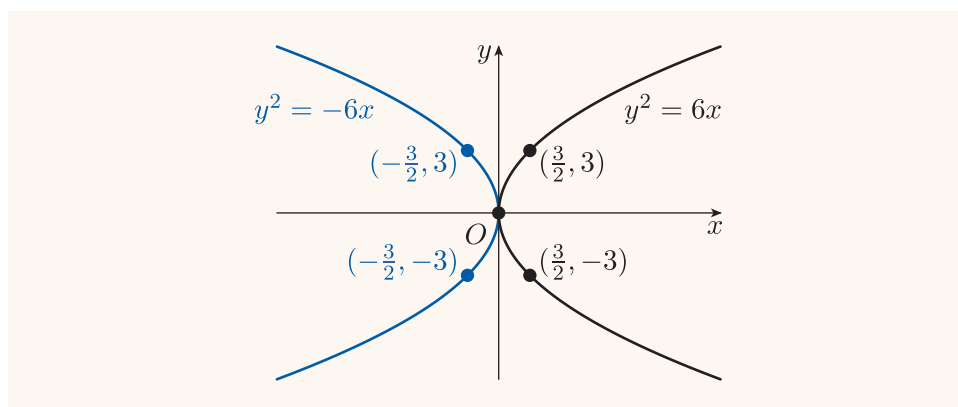


Figure 19 The parabolas with equations $y^2 = -6x$ (not in standard position) and $y^2 = 6x$ (in standard position)

2.4 Some practical applications of parabolas

In this short subsection you'll briefly meet some applications of parabolas. One of the most important practical features of a parabola is its reflection property.

When a ray of light strikes a surface, it is reflected at the same angle at which it struck, as shown in Figure 20(a). For a curved surface, the two angles are measured from a line that just touches the surface at the point where the ray strikes, as shown in Figure 20(b).

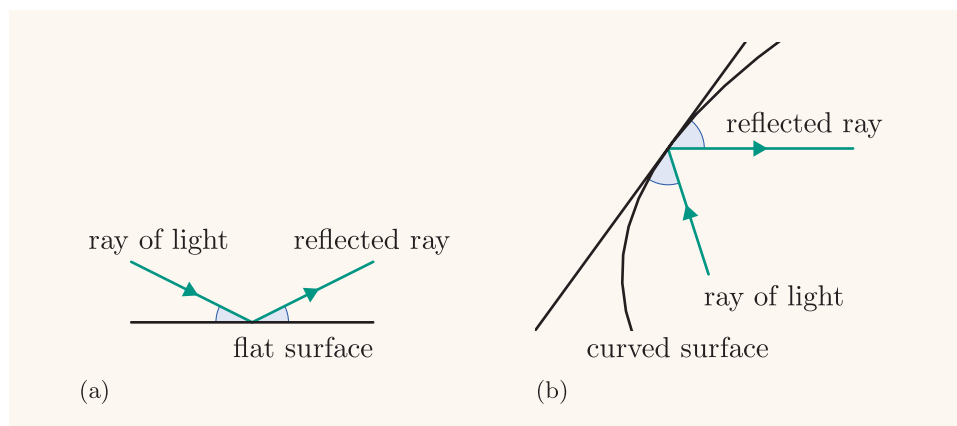


Figure 20 A ray of light reflected from (a) a flat surface (b) a curved surface

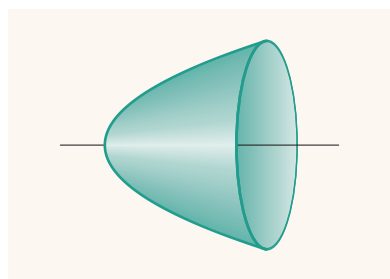


Figure 21 The shape of a parabolic reflector

A *parabolic reflector* is a reflective surface whose shape is formed by rotating part of a parabola through 360° about its axis of symmetry, as illustrated in Figure 21. Any cross-section of a parabolic reflector that passes through this axis of symmetry is therefore part of a parabola with the same focus and the same axis of symmetry as the original parabola. We refer to this point and line as the focus and axis of the parabolic reflector.

It can be proved, by using the focus–directrix property of a parabola, that if a light source is positioned at the focus of a parabolic reflector (with a surface designed to reflect light), then the rays of light are reflected in a direction that is parallel to the axis, as shown in Figure 22(a) for a cross-section of the reflector. For this reason, a torch usually has a parabolic reflector with the bulb at the focus.

Conversely, if rays of light that are parallel to the axis of a parabolic reflector strike its inner surface, then they are reflected to pass through its focus, as shown in Figure 22(b). This property is used in telescopes, satellite dishes and solar ovens (see Figure 23).

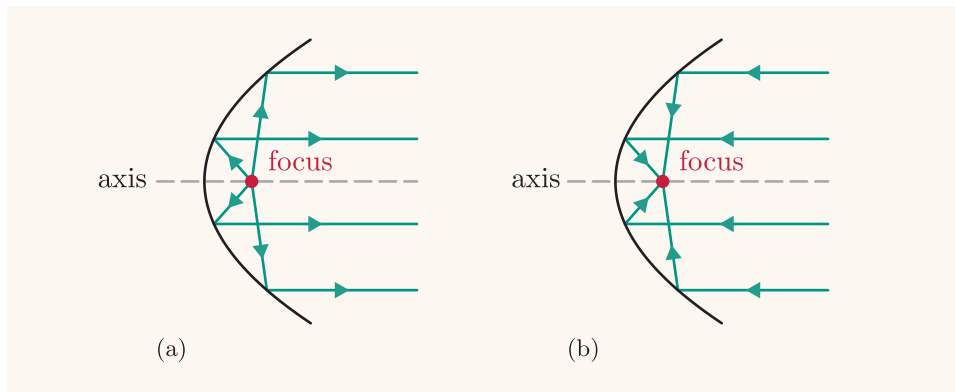


Figure 22 The reflection property of a parabola

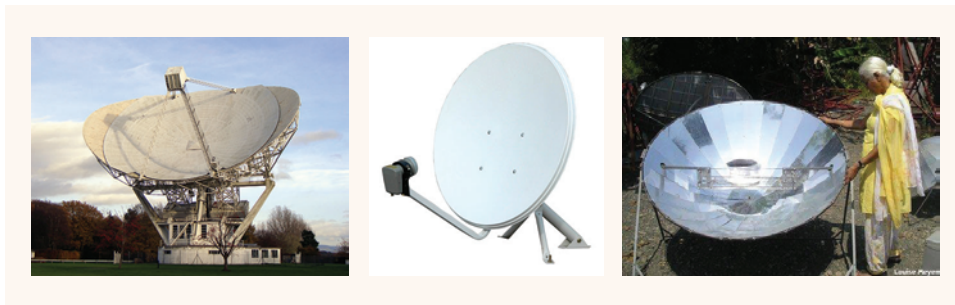


Figure 23 A radio telescope, a satellite dish and a solar oven

You've now seen three different ways to describe a parabola (further to the ways that you met in MST124 Unit 2):

- as a cross-section of a double cone
- as a set of points each of which is the same distance from a fixed point as it is from a fixed line
- for a parabola in standard position, as the set of points (x, y) that satisfy an equation of the form $y^2 = 4ax$, where $a > 0$.

In the next two subsections, you'll see how ellipses and hyperbolas can be described in ways similar to the second and third methods above.

3 Ellipses

In this section you'll see how an ellipse can be described in terms of distances from a point and a line, in a similar way to a parabola, and meet the general equation of an ellipse in standard position. You'll also learn about some properties of ellipses, and some of their practical applications.

3.1 The focus–directrix definition of an ellipse

In the last section, you learned that a parabola can be defined as the set of points P that satisfy the equation $PF = Pd$, where F is a fixed point called the *focus*, and d is a fixed line called the *directrix*, which does not pass through the focus.

In the next activity, you're asked to use an applet to explore the curves that are defined by equations such as $PF = \frac{1}{2}Pd$ and $PF = 3Pd$; that is, equations of the form $PF = ePd$, where e is a non-negative constant. Each point on such a curve is e times as far from the focus as it is from the directrix. You know that if you take $e = 1$ then you obtain a parabola, but you're asked to look at what happens when you take e to be some other non-negative constant.

The constant e is known as the **eccentricity** of the curve. Note that in this context e is simply a non-negative constant; there's no connection with the exponential constant 2.718...



Activity 7 Exploring curves given by equations of the form $PF = ePd$

Open the *Exploring focus–directrix definitions* applet.

It shows a point F (the focus), a line d (the directrix) and the curve formed by the points that satisfy the equation $PF = ePd$, where e takes a value that you can set. You can opt for the applet to show a particular point P on the curve, and its distances from F and d . You can move the focus F and the point P , and you can move and rotate the directrix d .

Make sure that the option to show both foci and directrices is *not* chosen. You won't need it in this activity, but it's mentioned later in this unit.

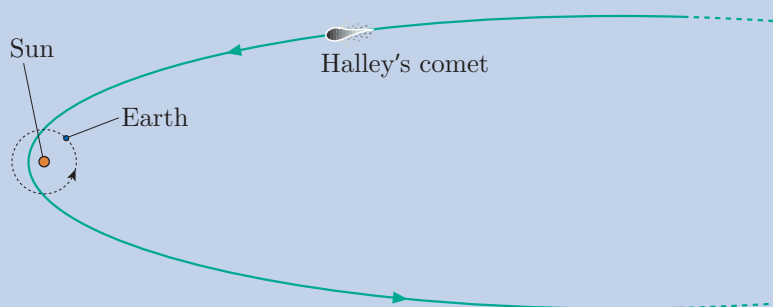
- (a) Gradually change the value of e and observe how the curve changes. What kind of curve (if any) do you obtain when e takes the following values?
 - (i) $e = 0$ (ii) $0 < e < 1$ (iii) $e = 1$ (iv) $e > 1$
- (b) Set $e = 0.5$. Drag the point P along the curve and check that it always seems to be half as far from F as it is from d .
- (c) Set $e = 2$. Drag the point P along the curve and check that it always seems to be twice as far from F as it is from d .

The type of curve defined by the equation $PF = ePd$ when $0 < e < 1$ is called an **ellipse**, and you'll learn more about it in this section. The type of curve, with two parts, defined by the equation $PF = ePd$ when $e > 1$ is called a **hyperbola**, and is the topic of Section 4.

Elliptical orbits

In 1609, Johannes Kepler, a mathematician, physicist and astronomer who was born and lived in an area that is now part of Germany, published his conclusion that the shapes of the orbits of the planets in our solar system are ellipses, with the Sun at a focus of each ellipse. He came to this conclusion, which is known as *Kepler's first law*, after studying the astronomical data then available, especially those from the Danish astronomer Tycho Brahe for the orbit of Mars.

Some eighty years later, Isaac Newton combined his law of universal gravitation with his laws of motion to show that any object moving under the gravitational attraction of the Sun alone has an orbit that is a conic, thereby generalising Kepler's first law. Further evidence for Newton's theory came in 1705 when Edmond Halley, an English mathematician and astronomer, correctly predicted that a particular comet, now known as Halley's comet, would return in 1758. Halley assumed that the comet followed an elliptical orbit, returning close to the Sun every 76 years.



Here's a summary of the focus-directrix definition of an ellipse, which we'll be using from now on.



Johannes Kepler (1571–1630)



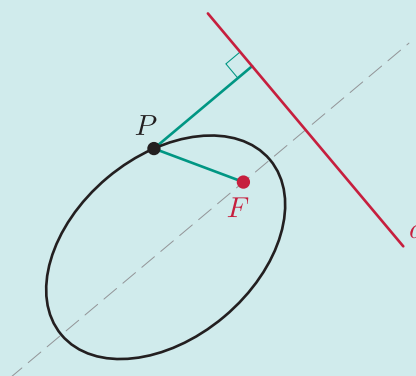
Edmond Halley (1656–1742)

The focus–directrix definition of an ellipse

Suppose that F is a point, d is a line that does not pass through F and e is a number such that $0 < e < 1$. Then the curve formed by the points P satisfying the equation

$$PF = e Pd$$

is an **ellipse** with **focus** F , **directrix** d and **eccentricity** e .



3.2 Ellipses in standard position

As with parabolas, it's complicated to find the equation of an ellipse that could be in any position in the plane. For this reason, in this unit we'll concentrate on ellipses that are positioned in the plane in a certain way that makes it straightforward to find and work with their equations.

Given any ellipse, positioned anywhere in the plane, you can first rotate it, along with its focus and directrix, until the directrix is vertical, with the focus to the left of the directrix. Then you can translate it vertically until the focus lies on the x -axis. Finally, you can translate it horizontally until the two points at which the ellipse crosses the x -axis are at equal distances on either side of the origin. This process is illustrated in Figure 24. The resulting ellipse is said to be in **standard position**. Every ellipse is congruent to an ellipse in standard position.

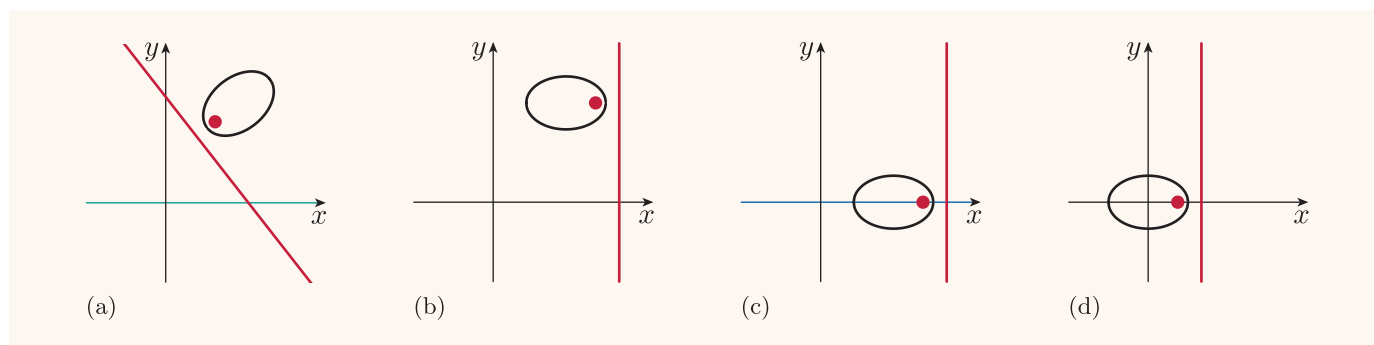


Figure 24 An ellipse rotated and translated into standard position

Let's now find a general equation for an ellipse in standard position. As you'll see, we need to use quite a lot of algebra to do this, but the final equation that we obtain is fairly simple.

Consider an ellipse in standard position, with focus F and directrix d , as shown in Figure 25, and eccentricity e . Let's denote the points where the ellipse crosses the x -axis by P and Q , with P being the right-hand point. Then, for some positive number a , the point P is $(a, 0)$ and the point Q is $(-a, 0)$. Also, let the focus be $(f, 0)$ and let the directrix be $x = g$. The x -values a , $-a$, f and g are marked along the x -axis in Figure 25.

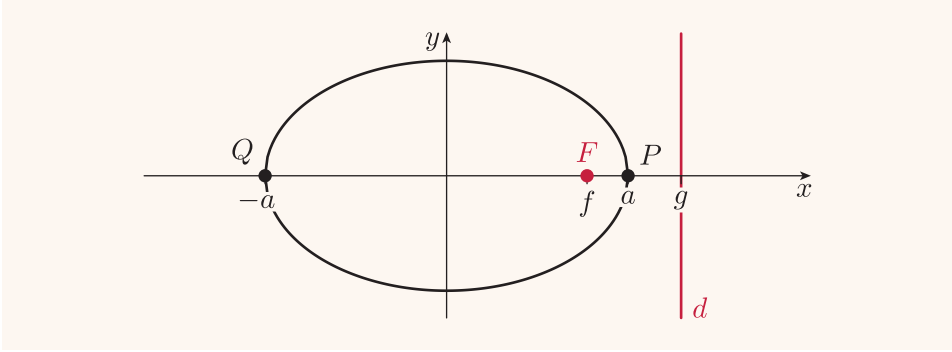


Figure 25 An ellipse in standard position, with focus F and directrix d

Let's start by finding expressions for the values f and g in terms of the positive x -intercept a and the eccentricity e .

From the diagram,

$$PF = a - f \quad \text{and} \quad Pd = g - a.$$

Since $PF = ePd$, we have

$$a - f = e(g - a). \tag{1}$$

Similarly,

$$QF = a + f \quad \text{and} \quad Qd = g + a.$$

Since $QF = eQd$, we have

$$a + f = e(g + a). \tag{2}$$

Adding equations (1) and (2) gives

$$2a = 2eg,$$

and hence

$$g = \frac{a}{e}.$$

Subtracting equation (1) from equation (2) gives

$$2f = 2ea,$$

and hence

$$f = ae.$$

So, in terms of a and e , the focus F is $(ae, 0)$ and the directrix d is $x = a/e$, as shown in Figure 26.

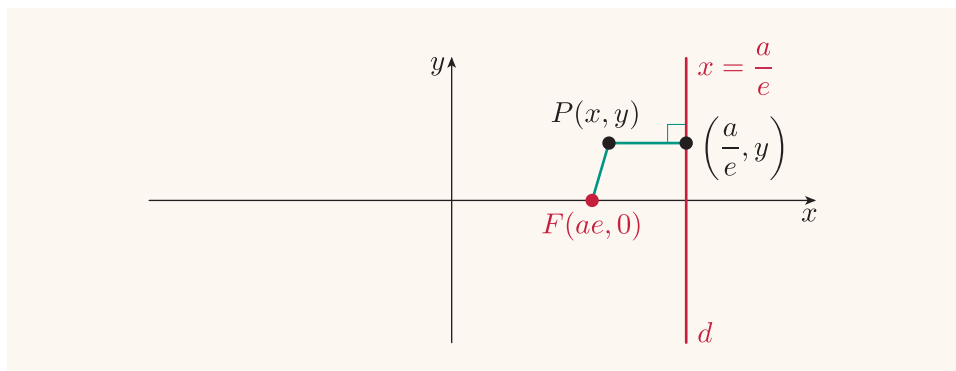


Figure 26 The focus $(ae, 0)$, the directrix $x = \frac{a}{e}$ and a point $P(x, y)$

Now let's use the coordinates $(ae, 0)$ of the focus and the equation $x = a/e$ of the directrix to find the equation of the ellipse in terms of a and e .

Let $P(x, y)$ be any point in the plane, as shown in Figure 26.

The distance of P from F is

$$PF = \sqrt{(x - ae)^2 + (y - 0)^2} = \sqrt{(x - ae)^2 + y^2}.$$

The distance of P from d is its distance from the point $(a/e, y)$, which is

$$Pd = \sqrt{\left(x - \frac{a}{e}\right)^2 + (y - y)^2} = \sqrt{\left(x - \frac{a}{e}\right)^2}.$$

The point P lies on the ellipse whenever it satisfies the equation

$$PF = ePd;$$

that is,

$$\sqrt{(x - ae)^2 + y^2} = e \sqrt{\left(x - \frac{a}{e}\right)^2}.$$

Taking the constant e on the right-hand side inside the square root sign gives

$$\sqrt{(x - ae)^2 + y^2} = \sqrt{e^2 \left(x - \frac{a}{e}\right)^2}$$

$$\sqrt{(x - ae)^2 + y^2} = \sqrt{\left(e \left(x - \frac{a}{e}\right)\right)^2}$$

$$\sqrt{(x - ae)^2 + y^2} = \sqrt{(ex - a)^2}.$$

Removing the square root signs and multiplying out the brackets gives

$$(x - ae)^2 + y^2 = (ex - a)^2$$

$$x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2$$

$$x^2 + a^2e^2 + y^2 = e^2x^2 + a^2.$$

Collecting all the terms in x and y on one side and all the other terms on the other side gives

$$\begin{aligned}x^2 - e^2x^2 + y^2 &= a^2 - a^2e^2 \\x^2(1 - e^2) + y^2 &= a^2(1 - e^2).\end{aligned}$$

Finally, dividing both sides by $a^2(1 - e^2)$ (which is non-zero, since $a > 0$ and $0 < e < 1$) gives

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \quad (3)$$

This is the equation of the ellipse in terms of a and e .

However, there's a more convenient way to write this equation. To obtain this alternative form, let's start by expressing the y -intercepts of the ellipse in terms of a and e . As you know, to find the y -intercepts of any curve you substitute $x = 0$ into its equation. Substituting $x = 0$ into equation (3) gives

$$\begin{aligned}\frac{0^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} &= 1 \\y^2 &= a^2(1 - e^2) \\y &= \pm\sqrt{a^2(1 - e^2)}.\end{aligned}$$

So the y -intercepts are

$$\sqrt{a^2(1 - e^2)} \quad \text{and} \quad -\sqrt{a^2(1 - e^2)}.$$

For conciseness, it's convenient to denote the value $\sqrt{a^2(1 - e^2)}$ by b . Then the y -intercepts are b and $-b$, as shown in Figure 27.

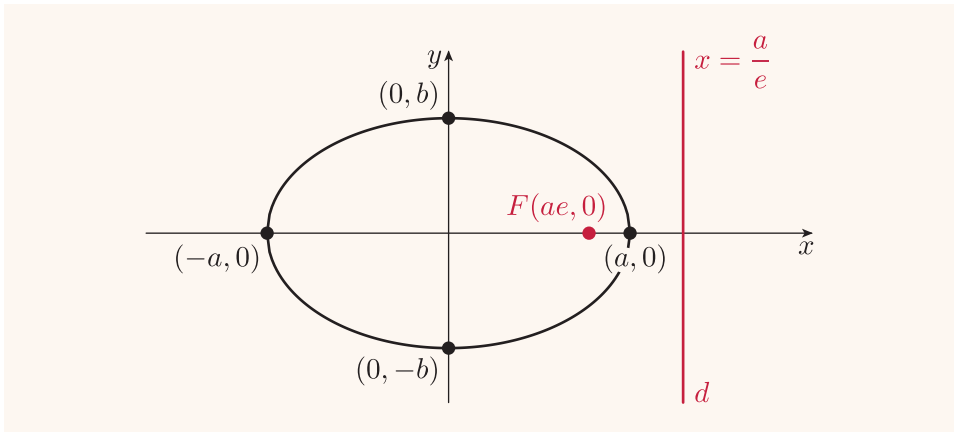


Figure 27 The points where an ellipse in standard position intersects the coordinate axes

The main advantage of defining the new constant b in this way is that, since $b^2 = a^2(1 - e^2)$, we can write equation (3) concisely as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is the standard equation for an ellipse in standard position.

The equation of an ellipse in standard position

The ellipse in standard position with eccentricity e (where $0 < e < 1$), focus $(ae, 0)$ and directrix $x = a/e$, where $a > 0$, has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where b is given by $b^2 = a^2(1 - e^2)$ and $b > 0$. The ellipse intersects the coordinate axes at $(\pm a, 0)$ and $(0, \pm b)$.

Note that since $b^2 = a^2(1 - e^2)$ where $0 < e < 1$, it follows that $b^2 < a^2$, and hence, since both a and b are positive, it follows that $b < a$. So an ellipse in standard position is wider than it is high.

If you know any two of the following five pieces of information about an ellipse in standard position, then you can use the facts in the box above to work out any of the other three pieces of information, and the equation of the ellipse:

the positive x -intercept a , the positive y -intercept b ,
the eccentricity e , the focus $(ae, 0)$, the directrix $x = \frac{a}{e}$.

Here's an example that illustrates this.



Example 3 Finding the equation of an ellipse in standard position

Find the equation of the ellipse in standard position with focus $(1, 0)$ and directrix $x = 4$.

Solution

Use the facts in the box above.

The ellipse in standard position that has eccentricity e and which intersects the x -axis at $(\pm a, 0)$ where $a > 0$ has focus $(ae, 0)$ and directrix $x = a/e$.

Compare these facts with the values given and hence find a , e and b .

Here, the focus is $(1, 0)$ and the directrix is $x = 4$.

So $ae = 1$ and $a/e = 4$.

Multiplying these equations together gives $a^2 = 4$. Since $a > 0$, this gives $a = 2$.

Substituting $a = 2$ into the equation $ae = 1$ gives $e = \frac{1}{2}$.

Substituting $a = 2$ and $e = \frac{1}{2}$ into the equation $b^2 = a^2(1 - e^2)$ gives

$$b^2 = 2^2 \left(1 - \left(\frac{1}{2}\right)^2\right) = 4 \times \frac{3}{4} = 3.$$

Since $b > 0$, this gives $b = \sqrt{3}$.

 Write down the equation of the ellipse, by using the values of a and b . 

So the equation of the ellipse is

$$\frac{x^2}{4} + \frac{y^2}{3} = 1.$$

Here's a similar example for you to try.

Activity 8 Finding the equation of an ellipse in standard position

Find the equation of the ellipse in standard position with focus $(3, 0)$ and directrix $x = 12$.

It's sometimes useful to work out the eccentricity e of an ellipse in standard position from its positive x - and y -intercepts a and b . You can obtain a general formula for e in terms of a and b by rearranging the equation $b^2 = a^2(1 - e^2)$ from the box on page 30, as follows:

$$\begin{aligned}\frac{b^2}{a^2} &= 1 - e^2 \\ e^2 &= 1 - \frac{b^2}{a^2}.\end{aligned}$$

Hence, since e is positive, the formula is as stated below.

The ellipse in standard position with equation $x^2/a^2 + y^2/b^2 = 1$ has eccentricity e given by

$$e = \sqrt{1 - \frac{b^2}{a^2}}.$$

Activity 9 Finding the eccentricity of an ellipse in standard position

What is the eccentricity of the ellipse with equation

$$\frac{x^2}{9} + \frac{y^2}{7} = 1?$$

3.3 Basic properties of ellipses

In this subsection we'll look at what the standard equation for an ellipse in standard position tells you about the shape of an ellipse.

Consider the ellipse in standard position with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Multiplying this equation through by a^2 gives

$$x^2 + \frac{a^2}{b^2}y^2 = a^2. \quad (4)$$

In this form, the equation resembles the equation $x^2 + y^2 = a^2$, which is the equation of the circle with centre the origin and radius a .

In fact there's a geometric connection between this circle and the ellipse, as follows. By equation (4), saying that a point (x, y) satisfies the equation of the ellipse is equivalent to saying that

$$x^2 + \left(\frac{ay}{b}\right)^2 = a^2.$$

This is in turn equivalent to saying that the point $(x, ay/b)$ satisfies the equation $x^2 + y^2 = a^2$ of the circle. So the point (x, y) lies on the ellipse whenever the point $(x, ay/b)$ lies on the circle, as shown in Figure 28.

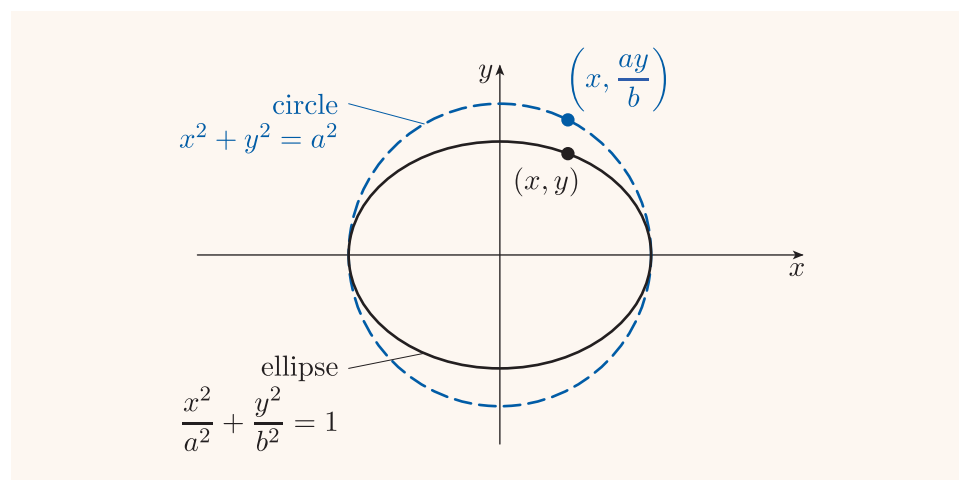


Figure 28 An ellipse in standard position and a related circle

Hence the circle can be obtained from the ellipse by

keeping the x -coordinate of each point on the ellipse unchanged and multiplying the corresponding y -coordinate by the factor a/b .

Equivalently, the ellipse can be obtained from the circle by

keeping the x -coordinate of each point on the circle unchanged and multiplying the corresponding y -coordinate by the factor $1/(a/b) = b/a$.

This shows that the ellipse is just the circle scaled by the factor b/a in the vertical direction. Since $0 < b/a < 1$, this scale factor squashes the ellipse in the vertical direction. The smaller is the value of b/a , the more squashed is the ellipse.

You can show, by using a similar argument, that the ellipse can alternatively be obtained from the circle with equation

$$x^2 + y^2 = b^2$$

by keeping the y -coordinate of each point on the circle unchanged and scaling the corresponding x -coordinate by the factor a/b , as shown in Figure 29. So you can think of an ellipse as either a squashed circle or a stretched circle.

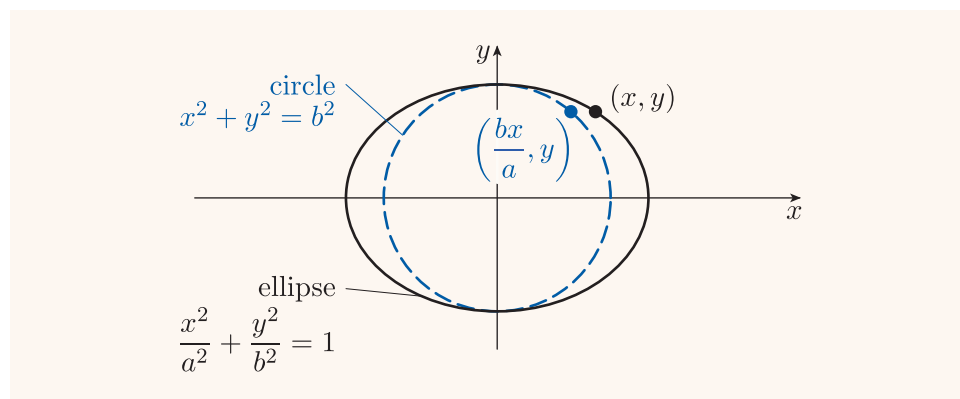


Figure 29 An ellipse in standard position and a related circle

When you used the *Exploring focus–directrix definitions* applet earlier in this section, you may have noticed that the amount by which an ellipse is squashed seems to depend on the value of its eccentricity e .

To see the connection between the values of b/a and e , you can rearrange the equation $b^2 = a^2(1 - e^2)$ from the box on page 30. Dividing through by a^2 and taking the square root of each side gives

$$\frac{b}{a} = \sqrt{1 - e^2}.$$

Hence, if the eccentricity e is close to 0, then the value of b/a is close to 1, so the ellipse is nearly circular. On the other hand, if the eccentricity e is close to 1, then the value of b/a is close to 0, so the ellipse is very squashed. Figure 30 shows three ellipses with different eccentricities.

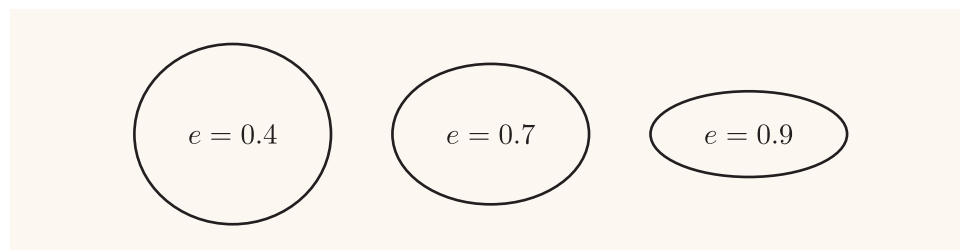


Figure 30 Three ellipses and their eccentricities

The Earth's orbit around the Sun is nearly circular, with eccentricity $e \approx 0.0167$, whereas the orbit of Halley's comet is elongated, with eccentricity $e \approx 0.967$.

Since an ellipse is a circle scaled in one direction, it has two axes of symmetry, at right angles to each other. For an ellipse in standard position, the axes of symmetry are the x -axis and the y -axis.

The point where the axes of symmetry of an ellipse cross is called its **centre**. The points where an ellipse crosses its axes of symmetry are called its **vertices**. Each of the two line segments that join two vertices on the same axis of symmetry of an ellipse is called an **axis** of the ellipse. The longer axis is called the **major axis** and the shorter axis is called the **minor axis**. For an ellipse in standard position, the major axis lies along the x -axis and the minor axis lies along the y -axis.

These definitions are illustrated in Figure 31.

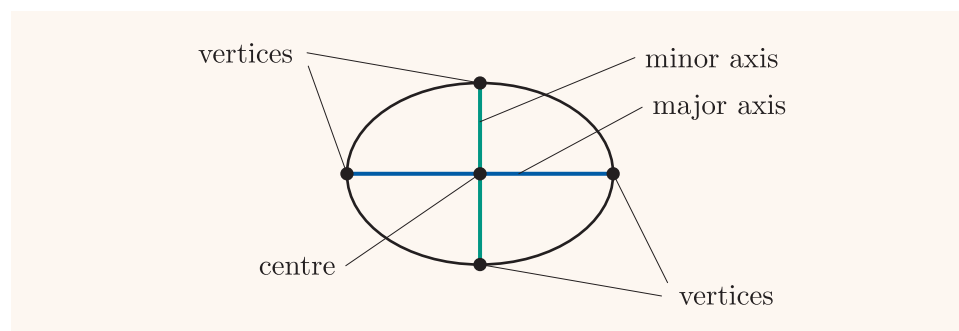


Figure 31 The centre, vertices and axes of an ellipse

The symmetry of an ellipse might lead you to suspect that it has another focus and another directrix, and this is indeed the case.

To see this, consider an ellipse in standard position, with focus F and directrix d , as shown in Figure 32, and eccentricity e . Let F_2 and d_2 be the reflections of F and d in the y -axis, as also shown in Figure 32.

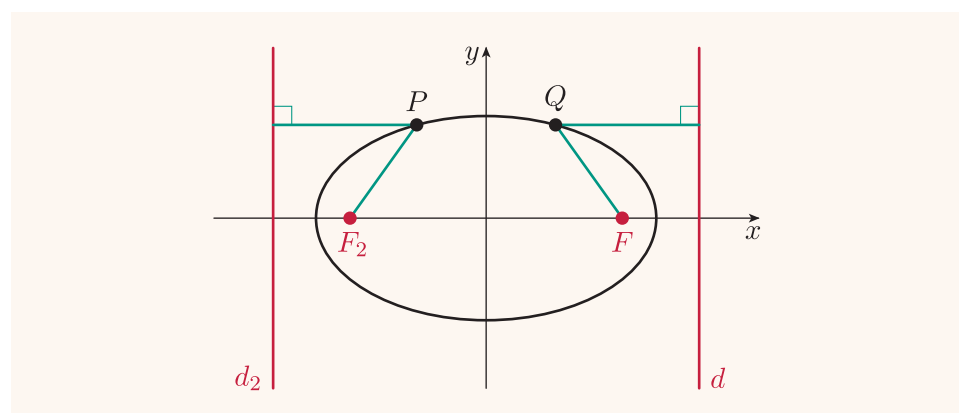


Figure 32 The foci and directrices of an ellipse in standard position

By the symmetry of the ellipse, for any point P on the ellipse, its reflection, Q say, in the y -axis also lies on the ellipse. Since the distance of Q from F is e times the distance of Q from d , it follows, again by the symmetry of the ellipse, that the distance of P from F_2 is e times the distance of P from d_2 . Hence F_2 is a focus and d_2 is a directrix of the ellipse.

The plurals of the words *focus* and *directrix* are *foci* and *directrices*, respectively. (The word ‘foci’ is pronounced ‘foh-see’ or ‘foh-kee’ in United Kingdom English.)

You might like to use the *Exploring focus-directrix definitions* applet to see the two foci and two directrices of some ellipses.

The following box summarises the main properties of an ellipse in standard position that you’ve met.

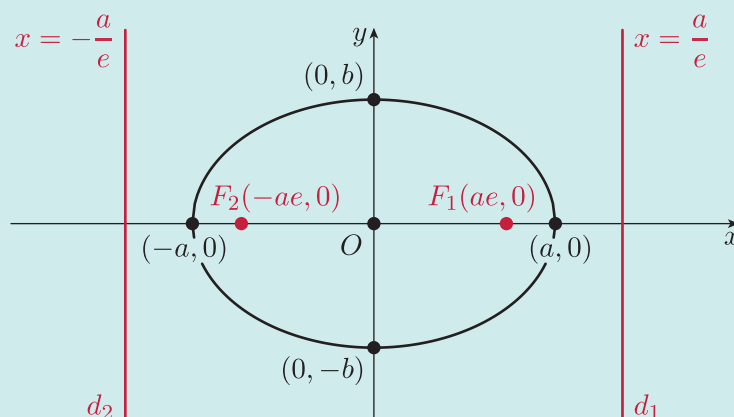
Properties of an ellipse in standard position

The ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } a > b > 0)$$

has the following properties.

- Its centre is the origin.
- Its vertices are $(\pm a, 0)$ and $(0, \pm b)$.
- Its axes of symmetry are the x - and y -axes.
- It can be obtained from the circle with equation $x^2 + y^2 = a^2$ by scaling it vertically by the factor b/a .
- Its major axis lies along the x -axis and its minor axis lies along the y -axis.
- Its eccentricity is $e = \sqrt{1 - \frac{b^2}{a^2}}$.
- Its foci are $(\pm ae, 0)$.
- Its directrices are $x = \pm \frac{a}{e}$.



If you have an equation that you can rearrange into the form $x^2/a^2 + y^2/b^2 = 1$, then you can recognise it as the equation of an ellipse in standard position, and use the properties in the box above to sketch this ellipse, as illustrated in the next example.

When you sketch an ellipse, you should draw it as a smooth curve that is symmetrical horizontally and vertically. You should label the vertices with their coordinates, and label the ellipse with its equation or include the equation in a caption. If you include the foci and directrices, then you should label the foci with their coordinates and the directrices with the their equations.

You may need to work out decimal approximations for some or all of the values a , b , ae and a/e to help you position the vertices, foci and directrices reasonably accurately, but you should label your sketch using the exact values if possible.

It can be tricky to draw an ellipse as a smooth, symmetrical curve by hand. A method that some students find helpful, not only for ellipses but for some other curves too, is to draw the curve *first*, before adding any of the other features such as the coordinate axes and the important points on the curve, such as vertices. Moving your pen or pencil fairly quickly when drawing the curve may also help you to make it smooth.



Example 4 Sketching an ellipse in standard position

Consider the equation

$$3x^2 + 4y^2 - 192 = 0.$$

- Show that this equation represents an ellipse in standard position.
- Find the vertices of the ellipse and sketch it.
- Find the eccentricity, foci and directrices of the ellipse, and add the foci and directrices to your sketch.

Solution

- The equation is

$$3x^2 + 4y^2 - 192 = 0.$$

🔍 Rearrange it into the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and identify the values of a and b . 🔍

Adding 192 to both sides and then dividing through by 192 gives

$$\frac{x^2}{64} + \frac{y^2}{48} = 1;$$

that is,

$$\frac{x^2}{8^2} + \frac{y^2}{(\sqrt{48})^2} = 1.$$

This equation is of the form $x^2/a^2 + y^2/b^2 = 1$, with $a = 8$ and $b = \sqrt{48} = 4\sqrt{3}$. Hence it is the equation of an ellipse in standard position.

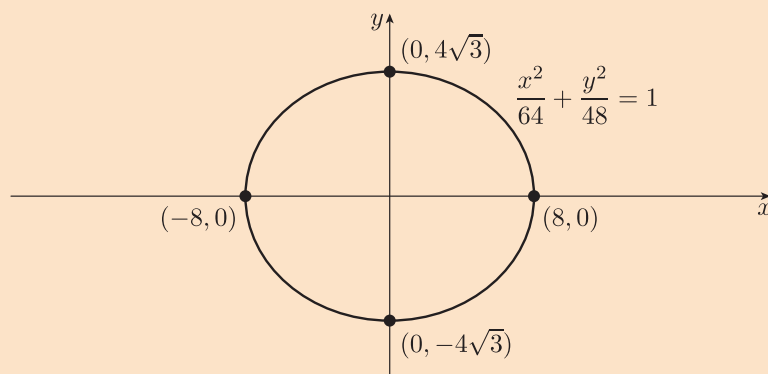
- (b)  To find the vertices, use the facts in the box above. Find decimal approximations as necessary to help with plotting. 

The vertices are $(\pm 8, 0)$ and $(0, \pm 4\sqrt{3})$.

The second pair is approximately $(0, \pm 6.9)$.

 Draw coordinate axes. Mark and label the vertices, using the same scale on both axes if reasonably possible, and hence draw the ellipse as a smooth curve. 

The ellipse is as shown below.



- (c)  To find the eccentricity, foci and directrices, again use the facts in the box above. 

The eccentricity is

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{48}{64}} = \sqrt{\frac{16}{64}} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

Hence

$$ae = 8 \times \frac{1}{2} = 4,$$

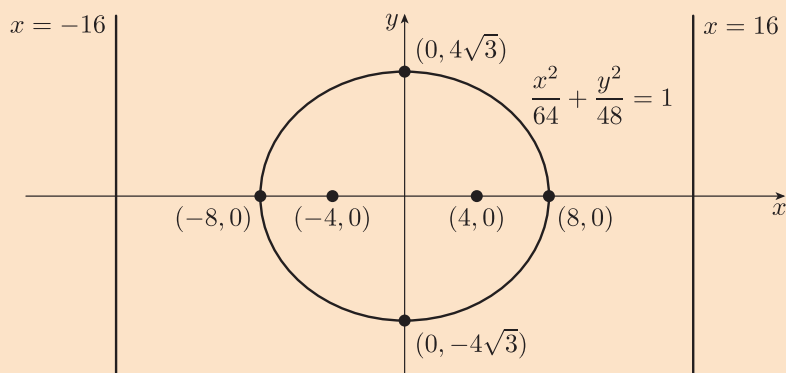
so the foci are $(\pm 4, 0)$.

Also

$$\frac{a}{e} = \frac{8}{1/2} = 16,$$

so the directrices are the lines $x = \pm 16$.

🧠 Draw and label the foci and directrices. 🧠



The next two activities are similar to Example 4 above. In these activities, and in many of the other activities in this unit, you'll find that some of the numerical values that you have to deal with are surds. In the solutions to the activities, the denominators of the surds are rationalised. For example, the surd $16/\sqrt{7}$ is written as $16\sqrt{7}/7$. As usual with surds, it isn't essential to rationalise the denominators. You can choose whether or not to do this, though you should always simplify the surds as much as possible otherwise. If you use your calculator for surds, you'll probably find that it rationalises denominators.

Activity 10 *Sketching an ellipse in standard position*

Consider the equation

$$9x^2 + 16y^2 = 144.$$

- Show that this equation represents an ellipse in standard position.
- Find the vertices of the ellipse and sketch it.
- Find the eccentricity, foci and directrices of the ellipse, and add the foci and directrices to your sketch.

Activity 11 *Sketching another ellipse in standard position*

Consider the equation

$$x^2 + 4y^2 - 4 = 0.$$

- Show that this equation represents an ellipse in standard position.
- Find the vertices of the ellipse and sketch it.
- Find the eccentricity, foci and directrices of the ellipse, and add the foci and directrices to your sketch.

You've seen that an equation of the form $x^2/a^2 + y^2/b^2 = 1$ where $a > b$ is the equation of an ellipse in standard position. An equation of the form $x^2/a^2 + y^2/b^2 = 1$ where $a < b$ is *not* the equation of an ellipse in standard position, but it's still the equation of an ellipse. This ellipse is obtained from the ellipse in standard position with equation $x^2/b^2 + y^2/a^2 = 1$ by swapping the x - and y -coordinates, so it's the reflection of this ellipse in the line $y = x$. For example, the equation $x^2/2 + y^2/5 = 1$ is the equation of the ellipse that's the reflection in the line $y = x$ of the ellipse in standard position with equation $x^2/5 + y^2/2 = 1$, as shown in Figure 33.

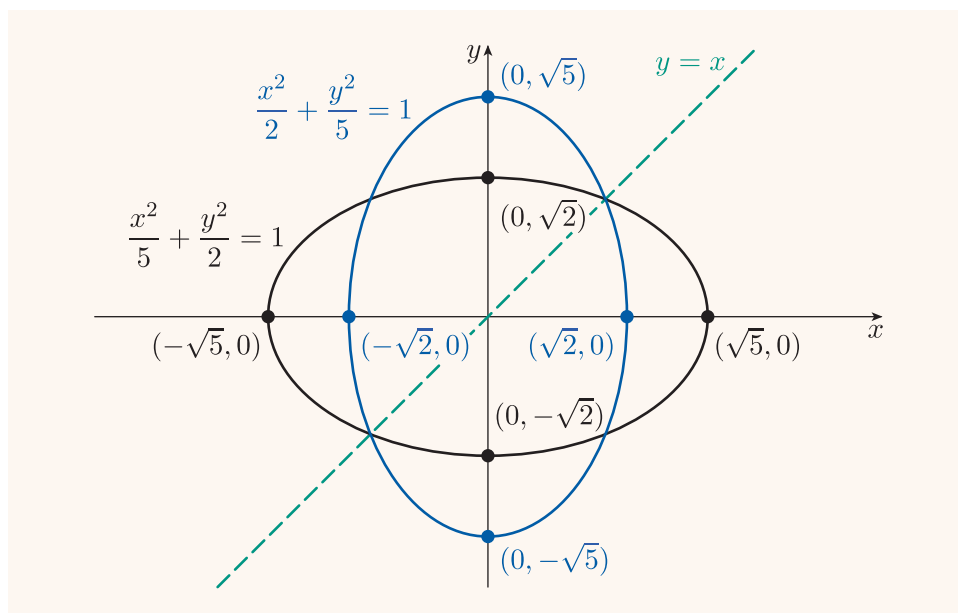


Figure 33 The ellipses with equations $x^2/2 + y^2/5 = 1$ (not in standard position) and $x^2/5 + y^2/2 = 1$ (in standard position)

So, in general, an equation of the form $x^2/a^2 + y^2/b^2 = 1$ where $a < b$ represents an ellipse with vertices $(\pm a, 0)$ and $(0, \pm b)$, exactly as in the case where $a > b$. You can sketch such an ellipse in the same way as an ellipse with $a > b$, by marking the vertices and drawing a smooth curve through them. You can try this in the next activity.

Activity 12 *Sketching an ellipse with a vertical major axis*

Rearrange the equation $16x^2 + 9y^2 = 144$ into the form $x^2/a^2 + y^2/b^2 = 1$, and sketch the ellipse that it represents.

You've now seen the type of curve that an equation of the form $x^2/a^2 + y^2/b^2 = 1$ represents if $a > b$ or $a < b$. If $a = b$, then the equation can be written as $x^2 + y^2 = a^2$, so it represents the circle whose centre is the origin and whose radius is a .

It's sometimes useful to think of a circle as a special type of ellipse, whose eccentricity is zero and whose foci coincide. However a circle doesn't have a focus-directrix property.

3.4 Some practical applications of ellipses

In this subsection you'll meet two properties of ellipses that have some practical applications.

The first property is that, for any ellipse, the distance from a point on the ellipse to a focus, plus the distance from the point to the other focus, as illustrated in Figure 34, is always the same no matter what point on the ellipse you choose. The total distance is always equal to the length of the major axis of the ellipse.

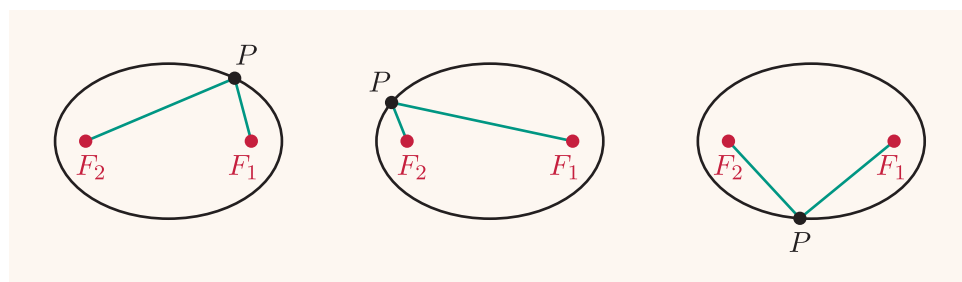


Figure 34 Distances between a point on an ellipse and the foci

This property is stated more formally as follows, for an ellipse in standard position.

If the point P lies on the ellipse with equation $x^2/a^2 + y^2/b^2 = 1$, then

$$PF_1 + PF_2 = 2a,$$

where F_1 and F_2 are the foci of the ellipse.

To see why this property holds, consider an ellipse with equation $x^2/a^2 + y^2/b^2 = 1$, and let its foci be F_1 and F_2 and its corresponding directrices be d_1 and d_2 , as shown in Figure 35.

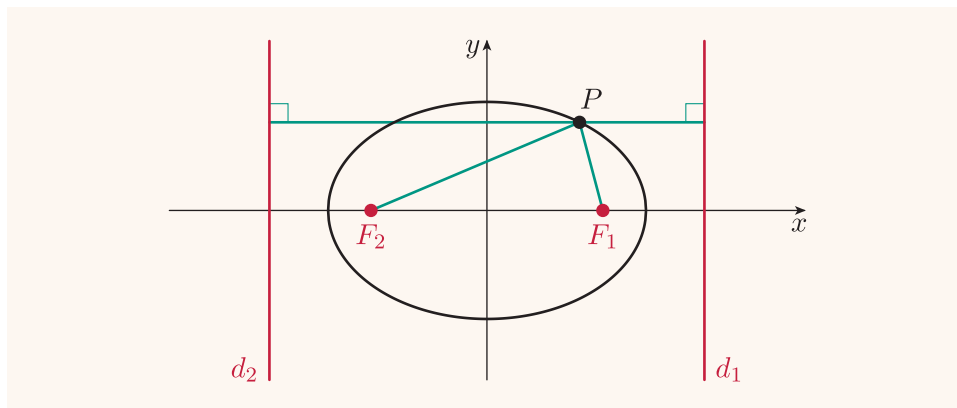


Figure 35 A point on an ellipse in standard position

Let P be a point on the ellipse. Then

$$PF_1 = ePd_1 \quad \text{and} \quad PF_2 = ePd_2,$$

where e is the eccentricity of the ellipse.

Adding these equations gives

$$PF_1 + PF_2 = e(Pd_1 + Pd_2).$$

The distance $Pd_1 + Pd_2$ is the distance between the two directrices of the ellipse, as you can see from Figure 35. Since the directrices have equations $x = a/e$ and $x = -a/e$, this distance is $2a/e$. Hence

$$PF_1 + PF_2 = e \times \frac{2a}{e} = 2a,$$

which is the property in the box.

It's also true that if you choose any two points F_1 and F_2 in the plane, and any positive number a , then the curve formed by all the points P such that $PF_1 + PF_2 = 2a$ is the ellipse whose foci are F_1 and F_2 and whose major axis has length $2a$. You can see intuitively that this is true by considering the property in the box above together with the 'gardener's method' described below. If you'd like to see a detailed proof, see the document *The two foci property of ellipses*, on the module website.

This fact gives you a simple method for drawing an ellipse. You stick two pins through a piece of paper on a corkboard, say, and attach the ends of a length of string to the pins. Then you use a pencil to hold the string taut, and move the pencil around, with the string always held taut, to trace out a curve, as shown in Figure 36. The curve that you obtain is an ellipse whose foci are the positions of the pins.

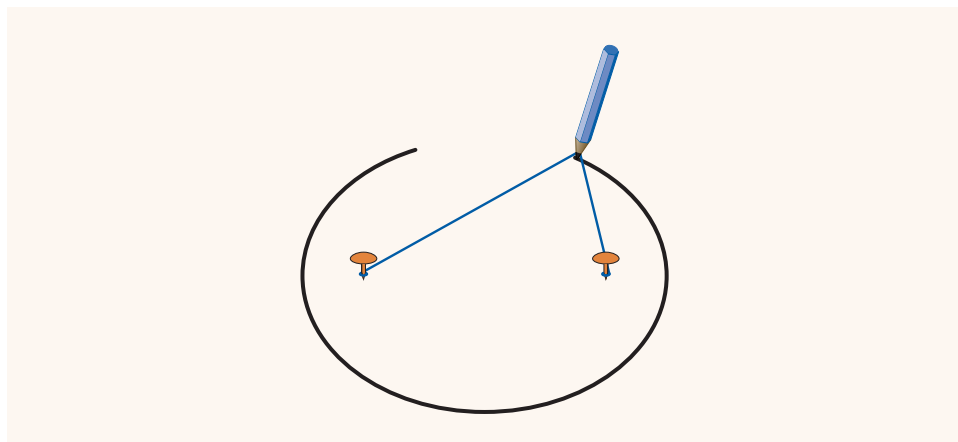


Figure 36 The gardener's method for drawing an ellipse

This method of drawing an ellipse is known as the *gardener's method*, since it's a convenient way to mark out elliptical flower beds. It has been known since at least the sixth century AD, when it was described by Anthemius, the architect of the great church of Hagia Sophia in Constantinople (now Istanbul).

Activity 13 Planning an elliptical flower bed

A garden design includes an elliptical flower bed 4 metres long and 3 metres wide. The ellipse is to be marked out by the gardener's method, using a length of string and two pegs.

- What length of string is needed? (Not including the extra length needed for the knots.)
- How far apart should the pegs be placed?

When the gardener's method is used in practice, it's usually easier not to attach the string to the two pins, but instead to make a loop of string and place it around the pins and pencil, as shown in Figure 37.

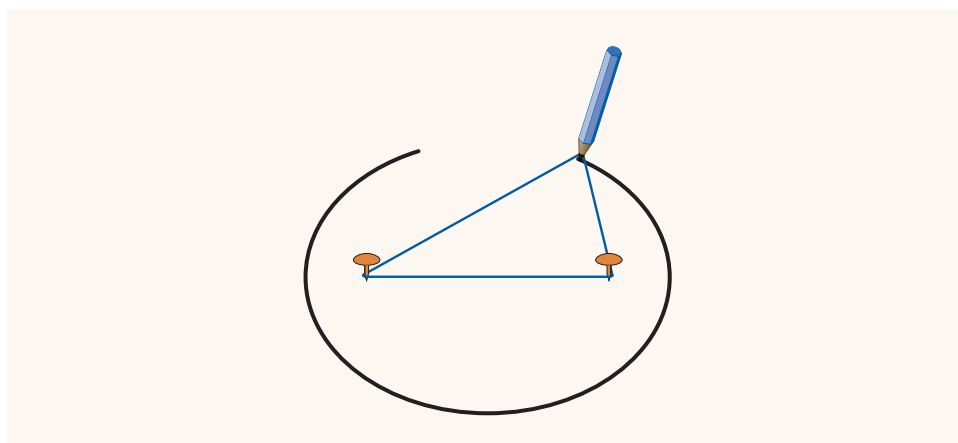


Figure 37 The gardener's method using a loop of string

Activity 14 *More planning for an elliptical flower bed*

If the elliptical flower bed in Activity 13 is to be marked out using a loop of string as described above, what length of loop is needed?

In 1984, the artist Chapman Kelley constructed two huge elliptical wildflower beds, covering over 6000 square metres, in Grant Park, Chicago, and named the work 'Wildflower Works'. It is pictured below. However in 2004 the Chicago Park District replaced the elliptical flower beds with two rectangular ones, covering less than half the original area. This led to two court cases in 2006 and 2011, with Kelley arguing that his artwork should be protected. He lost both cases.



The second property of ellipses that you'll meet in this subsection is a reflection property, similar to the one that you met for parabolas in the previous section.

An *elliptical reflector* (also known as an *ellipsoidal reflector*) is a reflective surface whose shape is formed by rotating all or part of an ellipse through 360° about its major axis. Any cross-section of an elliptical reflector that passes through this major axis is therefore part of an ellipse with the same foci as the original ellipse. We refer to these foci as the foci of the elliptical reflector.

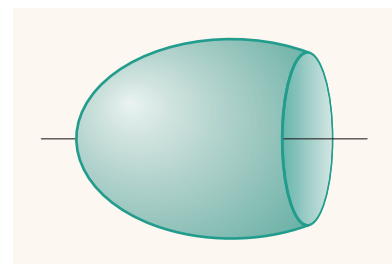


Figure 38 An elliptical reflector

It can be proved, by using the focus–directrix property of an ellipse, that if a light source is positioned at one of the foci of an elliptical reflector (with a surface designed to reflect light), then the rays of light are reflected to pass through the other focus, as shown in Figure 39 for a cross-section of an elliptical reflector.

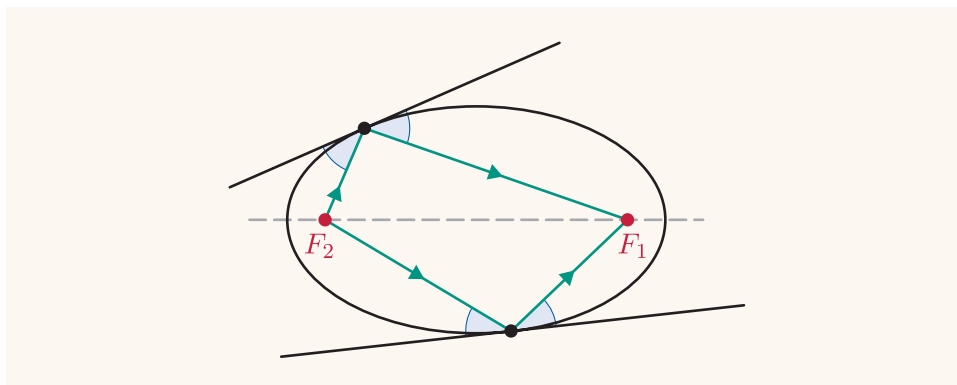
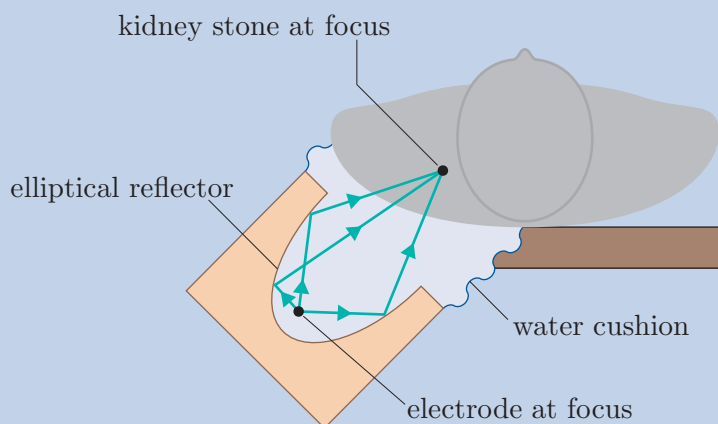


Figure 39 The reflection property of an ellipse

The reflection property of ellipses is used in the medical treatment of kidney stones by *lithotripsy*. A *lithotripter machine* is set up with its electrode at one focus of an elliptical reflector and the patient's kidney stone at the other, as shown below. The electrode emits a powerful shock wave that is reflected to the other focus and disintegrates the kidney stone.



4 Hyperbolas

In this section you'll see how the focus–directrix definition of a hyperbola, which you met briefly in Section 3, can be used to obtain a general equation for a hyperbola in standard position. You'll also learn about some properties of hyperbolas.

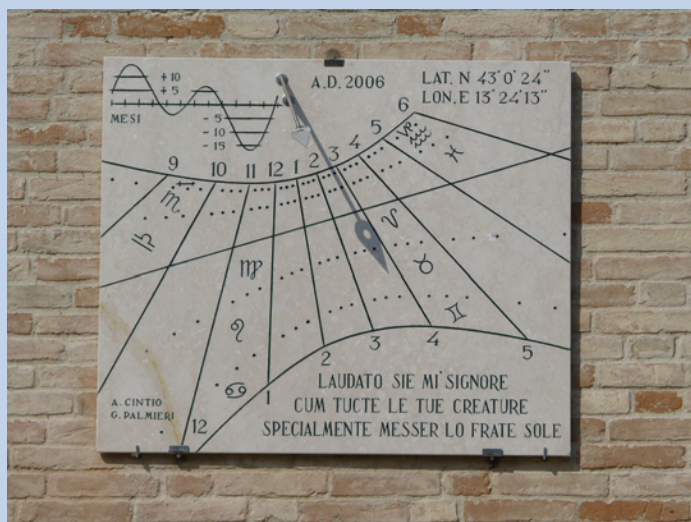
4.1 The focus–directrix definition of a hyperbola

In the previous two sections, you saw that if F is a fixed point and d is a fixed line that does not pass through F , then the points P that satisfy the equation $PF = e Pd$ form a parabola if $e = 1$, and an ellipse if $0 < e < 1$. Near the beginning of Section 3, you also saw that if $e > 1$, then the points form a curve called a *hyperbola*.

Like parabolas and ellipses, hyperbolas occur in many natural situations, such as in the motion of the shadow on a sundial, and in the motion of subatomic particles.

Sundials

Sundials have been used as a way of measuring time for over five thousand years. A sundial usually consists of a marked flat surface with a protruding thin stick, known as a *gnomon*. The sun creates a shadow of the gnomon on the surface, and the time of day can then be read off from the shadow, as illustrated here in a picture of the San Ruffino sundial in Italy.



The tip of the shadow of the gnomon on a sundial follows a path that is part of a hyperbola. You can find out more about this in the two-minute video clip *Sundials*, available on the module website.

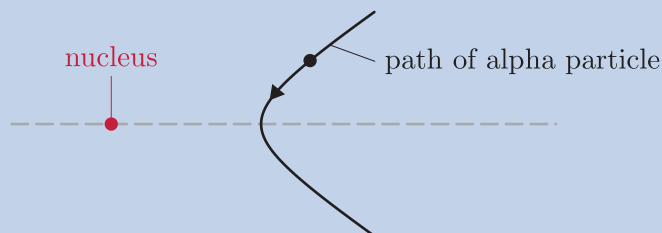




Ernest Rutherford
(1871–1937)

Motion of subatomic particles

In the early twentieth century, the physicist and Nobel Laureate Ernest Rutherford showed that the path of an alpha particle moving near the nucleus of an atom is part of a hyperbola, as shown below. The alpha particle is subject to an electrostatic repulsive force that pushes it away from the nucleus.



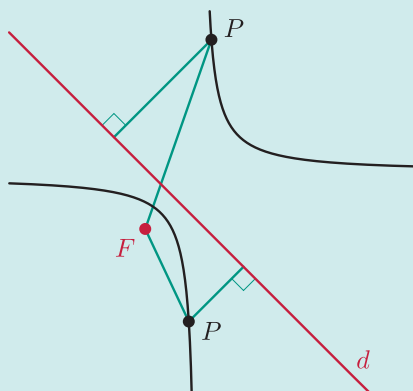
The focus–directrix definition of a hyperbola is summarised as follows.

Focus-directrix definition of a hyperbola

Suppose that F is a point, d is a line that does not pass through F , and e is a number such that $e > 1$. Then the curve formed by the points P satisfying the equation

$$PF = ePd$$

is a **hyperbola** with **focus** F , **directrix** d and **eccentricity** e .



4.2 Hyperbolas in standard position

As with parabolas and ellipses, in this section we'll consider hyperbolas that are positioned in the plane in a certain way that makes it straightforward to find and work with their equations. Given any hyperbola, positioned anywhere in the plane, you can first rotate it, along with its focus and directrix, until the directrix is vertical, with the focus to the right of the directrix. Then you can translate it vertically until the focus lies on the x -axis. Finally, you can translate it horizontally until the two points at which the hyperbola crosses the x -axis are at equal distances on either side of the origin.

This process is illustrated in Figure 24. The resulting hyperbola is said to be in **standard position**. Every hyperbola is congruent to a hyperbola in standard position.

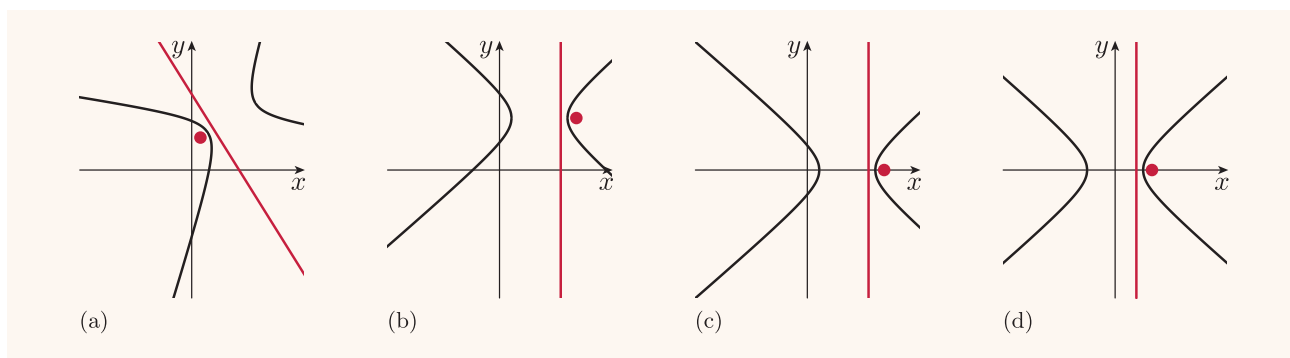


Figure 40 A hyperbola rotated and translated into standard position

We can find a general equation for a hyperbola in standard position by following the approach used for ellipses in Subsection 3.2. To do this, consider a hyperbola in standard position, with focus F and directrix d , as illustrated in Figure 41, and eccentricity e . Let's denote the points where the hyperbola crosses the x -axis by P and Q , with P being the right-hand point. Then, for some positive number a , the point P is $(a, 0)$ and the point Q is $(-a, 0)$. Also, let the focus be $(f, 0)$ and let the directrix be $x = g$. The x -values a , $-a$, f and g are marked along the x -axis in Figure 41.

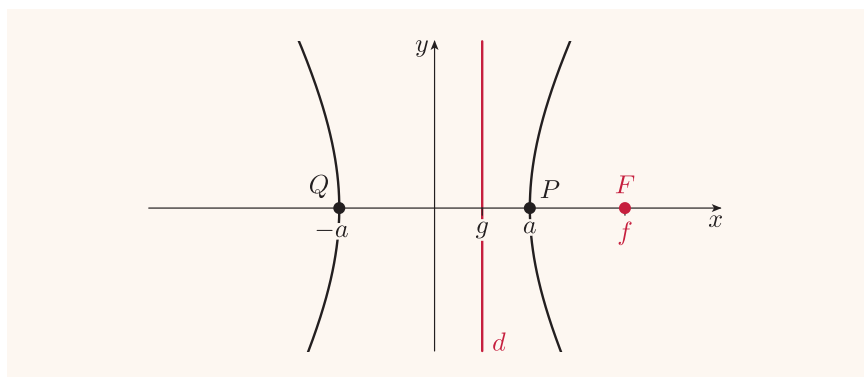


Figure 41 A hyperbola in standard position, with focus F and directrix d

In the next activity you're asked to find expressions for the values f and g in terms of the positive x -intercept a and the eccentricity e . You can do this by using a method similar to the one used for ellipses earlier, so you might find it helpful to look back at Subsection 3.2.

Activity 15 *Finding the focus and directrix of a hyperbola in standard position in terms of the positive x -intercept a and the eccentricity e*

- By considering the distances PF and Pd in Figure 41, and using the equation $PF = ePd$, find an equation that relates a , e , f and g .
- By considering the distances QF and Qd in Figure 41, and using the equation $QF = eQd$, find another equation that relates a , e , f and g .
- Use the equations that you found in parts (a) and (b) to express each of f and g in terms of a and e . Hence write down the coordinates of the focus F and the equation of the directrix d in terms of a and e .

In Activity 15 you should have found that the focus F has coordinates $(ae, 0)$ and the directrix d has equation $x = a/e$, as shown in Figure 42.

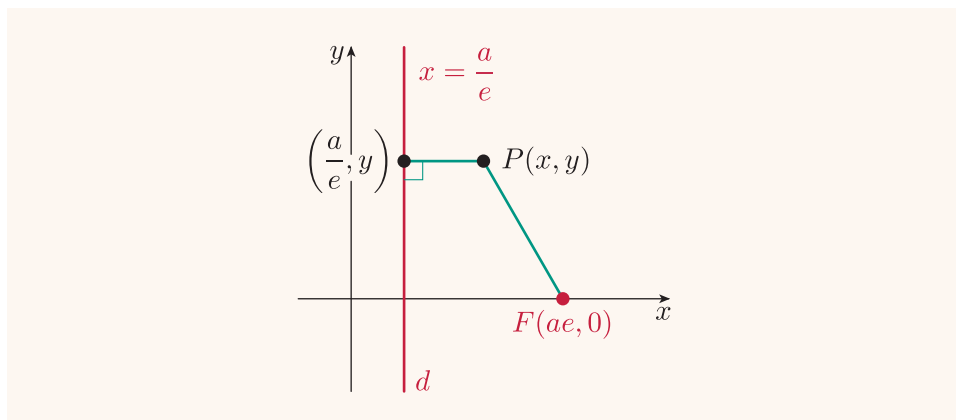


Figure 42 The focus and directrix of a hyperbola in standard position, and a general point P in the plane

So the coordinates of the focus and the equation of the directrix in terms of a and e are exactly the same as those found in Subsection 3.2 for the ellipse in standard position, namely $(ae, 0)$ and $x = a/e$. The defining equation of the hyperbola, $PF = ePd$, is also exactly the same as for the ellipse, so if you use the method of Subsection 3.2 to work out the equation of the hyperbola, then you obtain exactly the same answer as for the ellipse.

Hence the equation of the hyperbola in terms of a and e is the same as for the ellipse, namely

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \quad (5)$$

However, just as for an ellipse, there's a more helpful way to write the equation of a hyperbola, in terms of a and a new constant, b . For the ellipse, we took b to be the positive constant that satisfies the equation $b^2 = a^2(1 - e^2)$, but we can't do that for the hyperbola, because for the hyperbola we have $e > 1$ and hence $a^2(1 - e^2)$ is negative, which means that there's no such number b .

To get round this problem, we can rearrange equation (5) as

$$\frac{x^2}{a^2} - \frac{y^2}{-a^2(1 - e^2)} = 1;$$

that is,

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1.$$

Since $e > 1$, the expression $a^2(e^2 - 1)$ here is positive. So we can let b be the positive number that satisfies the equation $b^2 = a^2(e^2 - 1)$. Then the equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This is the standard equation for a hyperbola in standard position.

The equation of a hyperbola in standard position

The hyperbola in standard position with eccentricity e (where $e > 1$), focus $(ae, 0)$ and directrix $x = a/e$, where $a > 0$, has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where b is given by $b^2 = a^2(e^2 - 1)$ and $b > 0$. The hyperbola intersects the x -axis at $(\pm a, 0)$.

Unlike with ellipses, the values b and $-b$ aren't y -intercepts of the hyperbola with equation $x^2/a^2 - y^2/b^2 = 1$. The hyperbola doesn't intersect the y -axis at all, as you'll see confirmed in the next subsection. However the value of b affects the shape of the hyperbola in a similar way to the way that the value of b affects the shape of an ellipse in standard position, as you'll also see in the next subsection.

Unlike with the equation $b^2 = a^2(1 - e^2)$ that holds for ellipses in standard position, the equation $b^2 = a^2(e^2 - 1)$ that holds for hyperbolas in standard position doesn't impose any restrictions on the value of b compared to the value of a . For a hyperbola with equation $x^2/a^2 - y^2/b^2 = 1$, the value of b can be less than, equal to, or greater than the value of a .

As with ellipses, if you know any two of the following five pieces of information about a hyperbola in standard position with equation $x^2/a^2 - y^2/b^2 = 1$, then you can use the facts in the box above to work out any of the other three pieces of information, and the equation of the hyperbola:

the positive x -intercept a , the value b ,
the eccentricity e , the focus $(ae, 0)$, the directrix $x = \frac{a}{e}$.

In the next activity, you're given the focus and directrix of a hyperbola in standard position and asked to find its equation. You can do that by using a method similar to that of Example 3 and Activity 8 in Subsection 3.2, which are about ellipses.

Activity 16 Finding the equation of a hyperbola in standard position

Find the equation of the hyperbola in standard position with focus $(5, 0)$ and directrix $x = \frac{9}{5}$.

As with ellipses, it's sometimes useful to work out the eccentricity e of a hyperbola in standard position from the values of a and b in its equation $x^2/a^2 - y^2/b^2 = 1$. You can obtain a general formula for e in terms of a and b by rearranging the equation $b^2 = a^2(e^2 - 1)$ from the box on page 49, as follows:

$$\frac{b^2}{a^2} = e^2 - 1$$

$$e^2 = 1 + \frac{b^2}{a^2}.$$

Hence, since e is positive, the formula is as stated as follows.

The hyperbola in standard position with equation $x^2/a^2 - y^2/b^2 = 1$ has eccentricity e given by

$$e = \sqrt{1 + \frac{b^2}{a^2}}.$$

Activity 17 Finding the eccentricity of a hyperbola in standard position

What is the eccentricity of the hyperbola with equation

$$\frac{x^2}{9} - \frac{y^2}{7} = 1?$$

4.3 Basic properties of hyperbolas

In this subsection you'll look at what the standard equation for a hyperbola in standard position tells you about the shape of a hyperbola. As with parabolas and ellipses, this will not only tell you about hyperbolas in standard position, but about hyperbolas in general.

Consider a hyperbola in standard position, with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where a and b are positive numbers.

First, notice that if a particular point (x, y) satisfies the equation of the hyperbola, then the three points $(x, -y)$, $(-x, y)$ and $(-x, -y)$ also satisfy it, since $x^2 = (-x)^2$ and $y^2 = (-y)^2$ for any numbers x and y . Four such points are illustrated in Figure 43. Since the four points are reflections of each other in the x -axis and the y -axis, it follows that the x -axis and the y -axis are axes of symmetry of the hyperbola.

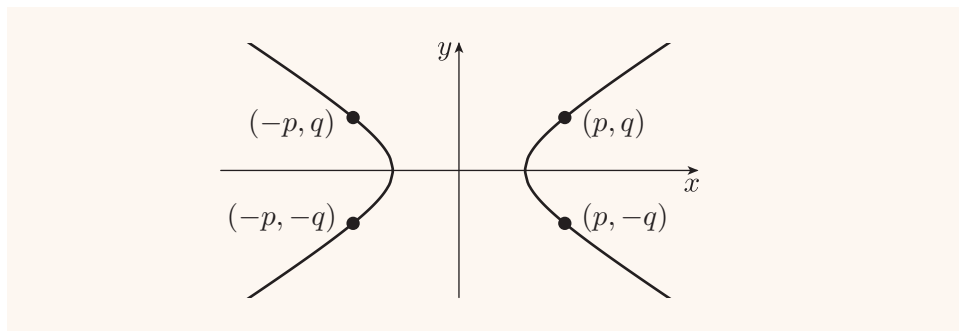


Figure 43 Four symmetrically positioned points on a hyperbola

We can determine some more properties of the hyperbola by rearranging its equation.

By rearranging the terms we obtain the equation

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1.$$

Then taking $1/a^2$ out as a factor on the right-hand side and multiplying through by b^2 gives

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2). \quad (6)$$

From this form of the equation you can see that no point (x, y) with $-a < x < a$ lies on the hyperbola, because for such a point the value of $x^2 - a^2$ is negative, and hence the value of $(b^2/a^2)(x^2 - a^2)$ is also negative and so can't be equal to the value of y^2 . In particular, the hyperbola has no y -intercepts.

We can determine an important property of the hyperbola by first writing equation (6) as two separate equations, each expressing y as a function of x :

$$y = \frac{b}{a}\sqrt{x^2 - a^2} \quad \text{and} \quad y = -\frac{b}{a}\sqrt{x^2 - a^2}.$$

The first of these two equations is the equation of the top half of the hyperbola (where $y \geq 0$), and the second equation is the equation of the bottom half (where $y \leq 0$).

Consider the equation of the top half of the hyperbola. Taking x^2 out as a factor from the expression inside the square root sign gives

$$y = \frac{b}{a}\sqrt{x^2 \left(1 - \frac{a^2}{x^2}\right)}.$$

For positive values of x , you can simplify this equation to

$$y = \frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}}. \quad (7)$$

So this is the equation of the top-right part of the hyperbola (where $x \geq 0$ and $y \geq 0$). Now think about values of x starting from $x = a$ and increasing. As x increases, the value of the positive number a^2/x^2 gets closer and closer to zero, so the value of $1 - a^2/x^2$ gets closer and closer to 1, while always remaining slightly smaller than 1. Hence the value of $\sqrt{1 - a^2/x^2}$ also gets closer and closer to 1, while always remaining slightly smaller than 1. So the value of y given by equation (7) gets closer and closer to the value of $(b/a)x$, while always remaining slightly smaller than $(b/a)x$. In other words, as x increases, the top-right part of the hyperbola gets closer and closer to the line $y = (b/a)x$, while always remaining slightly below it, as illustrated in Figure 44(a).

It follows, by the symmetry of the hyperbola, that its top-left, bottom-right and bottom-left parts all get closer and closer to one of the lines $y = (b/a)x$ and $y = -(b/a)x$ as the size of x increases, with the hyperbola lying between these lines, as illustrated in Figure 44(b).

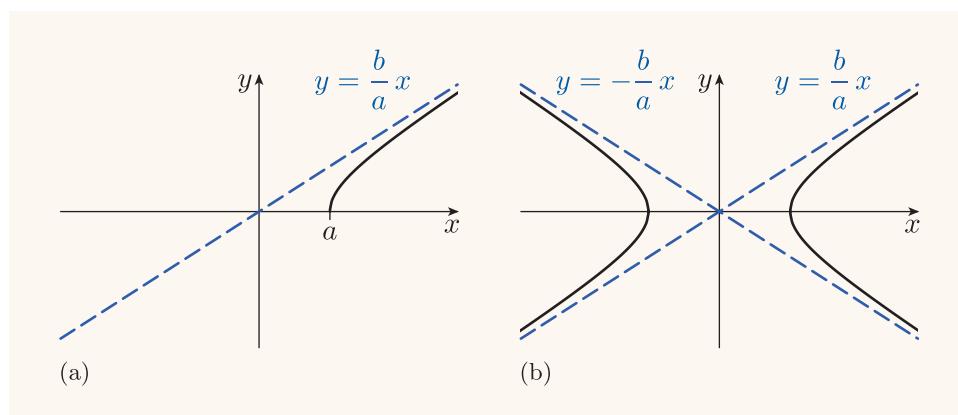


Figure 44 Lines approached by the hyperbola with equation $x^2/a^2 - y^2/b^2 = 1$

Remember that a line that a curve approaches more and more closely the further it is from the origin is called an **asymptote** of the curve. So the lines

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

are asymptotes of the hyperbola.

The top-right and bottom-right parts of the hyperbola both contain the point $(a, 0)$, so they're joined at this point, to form one continuous part of the hyperbola. Similarly, the top-left and bottom-left parts of the hyperbola both contain the point $(-a, 0)$, so they're joined at this point, to form the other continuous part of the hyperbola. The two continuous parts of a hyperbola are known as its **branches**.

Since the hyperbola lies between its asymptotes $y = \pm(b/a)x$, the smaller the value of b/a is, the more squashed the hyperbola is vertically.

As for an ellipse, the value of the eccentricity e of a hyperbola tells you how squashed the hyperbola is. To see the connection between the values of b/a and e for a hyperbola in standard position, you can rearrange the equation $b^2 = a^2(e^2 - 1)$ from the box on page 49. Dividing through by a^2 and taking the square root of each side gives

$$\frac{b}{a} = \sqrt{e^2 - 1}.$$

Hence, if the eccentricity e is close to 1, then the value of b/a is close to 0, so the hyperbola is very squashed. On the other hand, if the eccentricity e is large, then so is b/a , so the hyperbola is stretched out. Figure 45 shows three hyperbolas with different eccentricities.

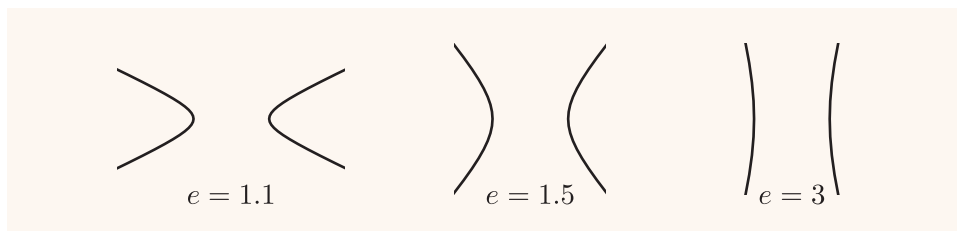


Figure 45 Three hyperbolas and their eccentricities

If the values a and b in the equation $x^2/a^2 - y^2/b^2 = 1$ of a hyperbola in standard position are such that $a = b$, then the equation of the hyperbola can be written as $x^2 - y^2 = a^2$, and its asymptotes are $y = \pm x$, which are perpendicular lines. A hyperbola whose asymptotes are perpendicular is called a **rectangular hyperbola**. An example of a rectangular hyperbola is shown in Figure 46.

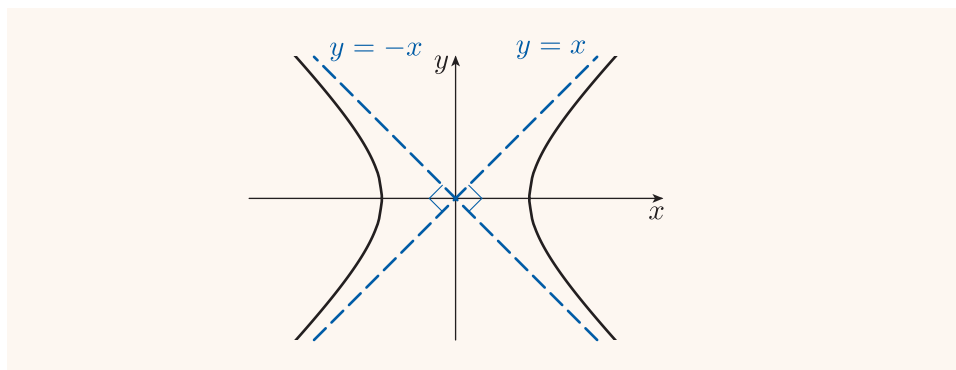


Figure 46 A rectangular hyperbola in standard position

The point where the axes of symmetry of a hyperbola cross (which is also the point where its asymptotes cross) is called its **centre**. The points where a hyperbola crosses an axis of symmetry are called its **vertices**. For a hyperbola in standard position, its centre is the origin and its vertices lie on the x -axis.

As with an ellipse, the symmetry of a hyperbola might lead you to suspect that it has another focus and another directrix, and this is indeed the case. To see this, consider a hyperbola in standard position, with focus F and directrix d , as shown in Figure 47, and eccentricity e . Let F_2 and d_2 be the reflections of F and d in the y -axis, as also shown in Figure 47. By the symmetry of the hyperbola, for any point P on the hyperbola, its reflection, Q say, in the y -axis also lies on the hyperbola. Since the distance of Q from F is e times the distance of Q from d , it follows, again by the symmetry of the hyperbola, that the distance of P from F_2 is e times the distance of P from d_2 . Hence F_2 is a focus and d_2 is a directrix of the hyperbola.

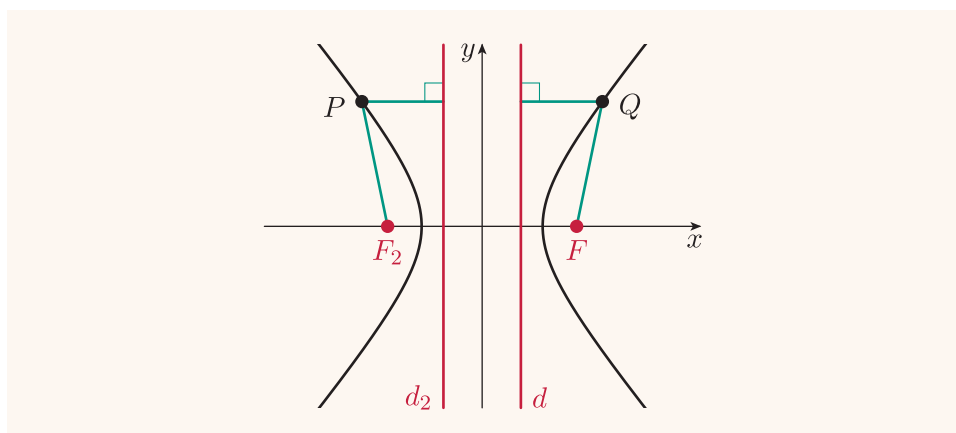


Figure 47 The foci and directrices of a hyperbola in standard position

You might like to use the *Exploring focus-directrix definitions* applet to see the two foci and two directrices of some hyperbolas.

The following box summarises the properties of a hyperbola in standard position that you've met.

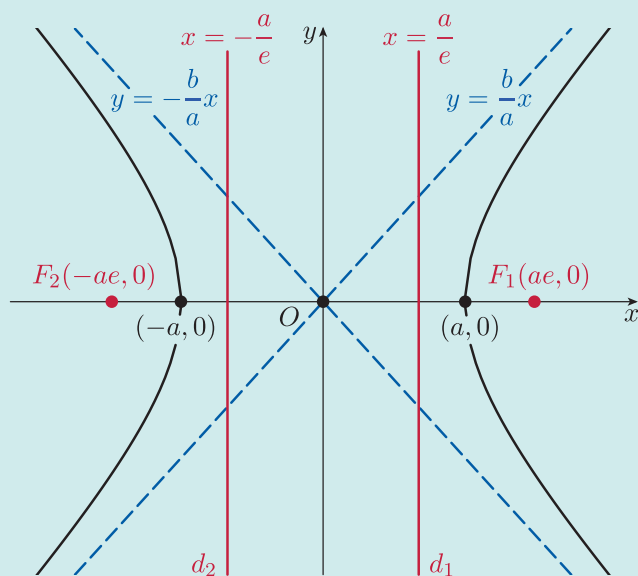
Properties of a hyperbola in standard position

The hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (\text{where } a, b > 0)$$

has the following properties:

- its centre is the origin
- its vertices are $(\pm a, 0)$
- its axes of symmetry are the x - and y -axes
- its asymptotes are $y = \pm \frac{b}{a}x$
- its eccentricity is $e = \sqrt{1 + \frac{b^2}{a^2}}$
- its foci are $(\pm ae, 0)$
- its directrices are $x = \pm \frac{a}{e}$
- it consists of two unbounded branches, one with $x \geq a$ and the other with $x \leq -a$.



If you have an equation that you can rearrange into the form $x^2/a^2 - y^2/b^2 = 1$, then you can recognise it as the equation of a hyperbola in standard position, and use the properties in the box above to help you sketch this hyperbola, as illustrated in the next example.

When you sketch a hyperbola, you should draw it as two smooth curves that are symmetrical horizontally and vertically, and that approach the asymptotes for large positive and negative values of x . Take care that the curves approach the asymptotes rather than curving away from them. You should label the vertices with their coordinates and the asymptotes with their equations, and you should label the hyperbola with its equation or include the equation in a caption. If you include the foci and directrices, then you should label the foci with their coordinates and the directrices with their equations, if possible.

You may need to work out decimal approximations for some or all of the values a , b/a , ae and a/e to help you position the vertices, asymptotes, foci and directrices reasonably accurately, but you should label your sketch using the exact values.



Example 5 *Sketching a hyperbola in standard position*

Consider the equation

$$x^2 - 9y^2 = 9.$$

- Show that this equation represents an hyperbola in standard position.
- Find the vertices and asymptotes of the hyperbola and sketch the hyperbola.
- Find the eccentricity, foci and directrices of the hyperbola, and add the foci and directrices to your sketch.

Solution

- The equation is

$$x^2 - 9y^2 = 9.$$

🔍 Rearrange it into the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and identify the values of a and b . 🔍

Dividing through by 9 gives

$$\frac{x^2}{9} - y^2 = 1;$$

that is,

$$\frac{x^2}{3^2} - \frac{y^2}{1^2} = 1.$$

This equation is of the form $x^2/a^2 - y^2/b^2 = 1$, with $a = 3$ and $b = 1$. Hence it is the equation of a hyperbola in standard position.

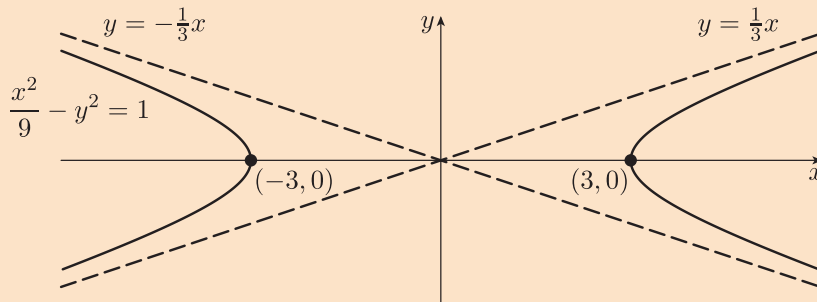
- (b) To find the vertices and asymptotes, use the facts in the box above.

The vertices are $(\pm 3, 0)$.

The asymptotes are $y = \pm \frac{1}{3}x$.

Draw coordinate axes, and draw and label the vertices and asymptotes. Then sketch the hyperbola as a smooth curve.

The hyperbola is as follows.



- (c) To find the eccentricity, foci and directrices, again use the facts in the box above.

The eccentricity is

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{9}} = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3}.$$

Hence

$$ae = 3 \times \frac{\sqrt{10}}{3} = \sqrt{10},$$

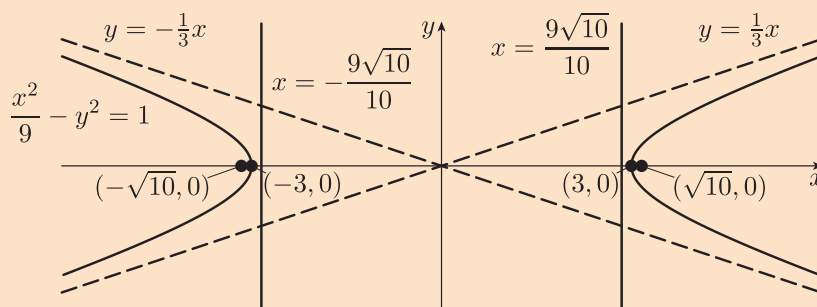
so the foci are $(\pm\sqrt{10}, 0)$; that is, approximately $(\pm 3.2, 0)$.

Also

$$\frac{a}{e} = \frac{3}{\sqrt{10}/3} = \frac{9}{\sqrt{10}} = \frac{9\sqrt{10}}{10},$$

so the directrices are the lines $x = \pm \frac{9\sqrt{10}}{10}$; that is, approximately $x = \pm 2.8$.

Draw and label the foci and directrices.



An alternative way to find the equations of the asymptotes in Example 5, rather than using the general equations $y = \pm(b/a)x$, is to rearrange the equation of the hyperbola into the form $y^2 = \frac{1}{9}x^2 - 1$. For large positive and negative values of x , the value of the constant -1 on the right-hand side of this equation is insignificant compared to the value of x^2 , so $y^2 \approx \frac{1}{9}x^2$ and hence $y \approx \pm\frac{1}{3}x$. This tells you that the equations of the asymptotes are $y = \pm\frac{1}{3}x$. You can use this method to find the asymptotes of any hyperbola from its equation.

Activity 18 Sketching a hyperbola in standard position

Consider the equation

$$9x^2 - 16y^2 = 144.$$

- Show that this equation represents an hyperbola in standard position.
- Find the vertices and asymptotes of the hyperbola and sketch the hyperbola.
- Find the eccentricity, foci and directrices of the hyperbola, and add the foci and directrices to your sketch.

Activity 19 Sketching another hyperbola in standard position

Repeat Activity 18 for the equation

$$4x^2 - 3y^2 = 1.$$

In the final activity of this section, you'll need to use what you've learned about the equations of parabolas, ellipses and hyperbolas in standard position.

Activity 20 *Identifying types of conics in standard position from their equations*

Rearrange each of the following equations into the standard form for the equation of a conic in standard position, identify the type of conic and sketch the conic.

(You are not asked to include the foci and directrices of the conics in your sketches.)

(a) $4y^2 - 25x^2 + 1 = 0$ (b) $3x - y^2 = 0$ (c) $x^2 + 18y^2 - 9 = 0$

The following box summarises some of the main properties of conics in standard position. Note in particular the similarities between the foci and directrices of the three types of conic, and also the different value ranges and expressions for the eccentricity e .

Properties of ellipses, parabolas and hyperbolas in standard position			
Curve	Ellipse	Parabola	Hyperbola
Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where $a > b > 0$)	$y^2 = 4ax$ (where $a > 0$)	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (where $a, b > 0$)
Vertices	$(\pm a, 0), (0, \pm b)$	$(0, 0)$	$(\pm a, 0)$
Asymptotes			$y = \pm \frac{b}{a} x$
Eccentricity	$0 < e < 1$ $e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = 1$	$e > 1$ $e = \sqrt{1 + \frac{b^2}{a^2}}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$
Directrices	$x = \pm \frac{a}{e}$	$x = -a$	$x = \pm \frac{a}{e}$

5 Conics not in standard position

In this unit you've met the equations of conics in standard position. You've seen that by using these equations you can deduce properties of conics in general.

It's possible to find the equations of conics that aren't in standard position by using the fact that any conic can be rotated and translated to give a conic in standard position. The techniques that you need to use to find such equations are beyond the scope of this module, but you can learn about them in higher-level modules. For now, it's useful for you to at least be aware of the forms taken by the equations of conics that aren't in standard position.

In MST124 Unit 5 you saw that every circle, no matter what its position in the plane, has an equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where A , C , D , E and F are constants and $A = C$. For example, the circle with centre $(2, 1)$ and radius 3, shown in Figure 48, has equation

$$x^2 + y^2 - 4x - 2y - 4 = 0.$$

This equation comes from multiplying out the equation

$$(x - 2)^2 + (y - 1)^2 = 3^2.$$

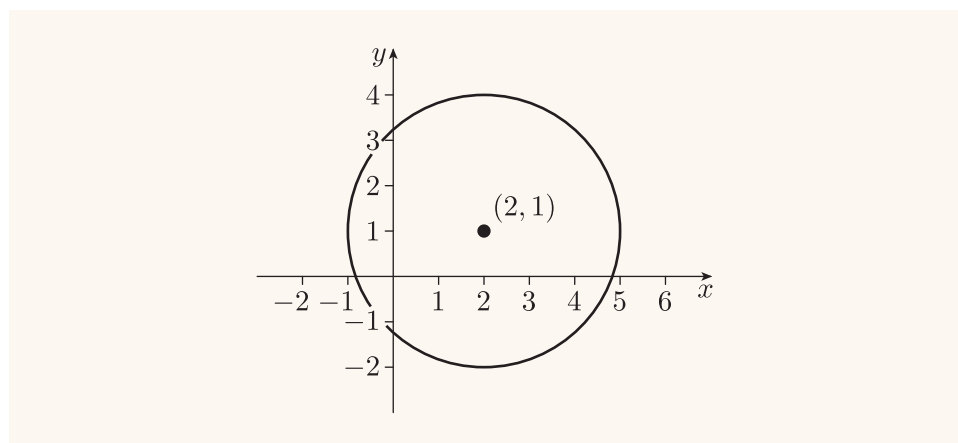


Figure 48 The circle with equation $x^2 + y^2 - 4x - 2y - 4 = 0$

It turns out that every conic, no matter what its position in the plane, has an equation of the same form as the equation of a circle, except that the equation may also include a term in xy , and the coefficients of x^2 and y^2 may be different from each other. In other words, the following fact holds.

General equation for a conic

Every conic, no matter what its position in the plane, has an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A , B , C , D , E and F are constants.

For example, the ellipse in Figure 49(b) has equation

$$4x^2 + 9y^2 - 16x - 18y - 11 = 0. \quad (8)$$

This ellipse is obtained from the ellipse in standard position with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

which is shown in Figure 49(a), by translating it 2 units right and 1 unit up.

(In fact, in a similar way to the equation above for the circle in Figure 48, equation (8) comes from multiplying out the equation

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1.$$

but you're not expected to deal with such manipulations in this module.)

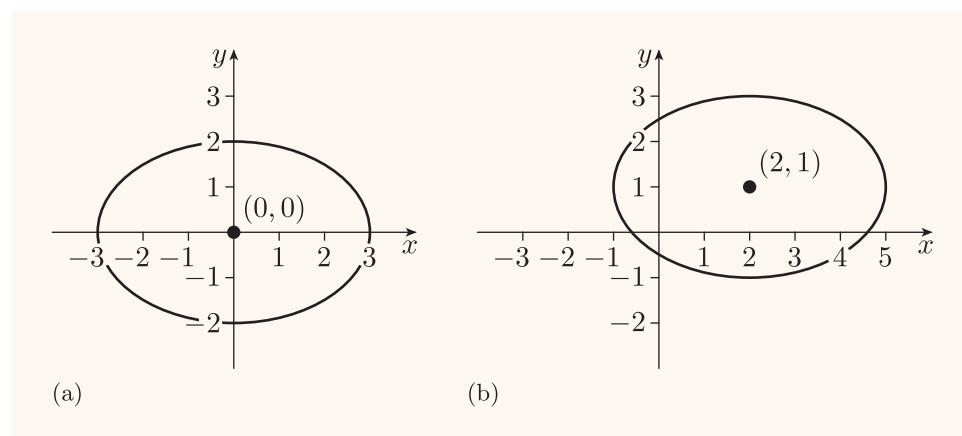


Figure 49 Two congruent ellipses

As another example, the rectangular hyperbola in Figure 50(b) has equation $xy - 1 = 0$. You're probably more familiar with this equation in the form $y = 1/x$. This hyperbola is obtained from the rectangular hyperbola in standard position with equation $x^2 - y^2 = 2$, which is shown in Figure 50(a), by rotating it anticlockwise about the origin through 45° .

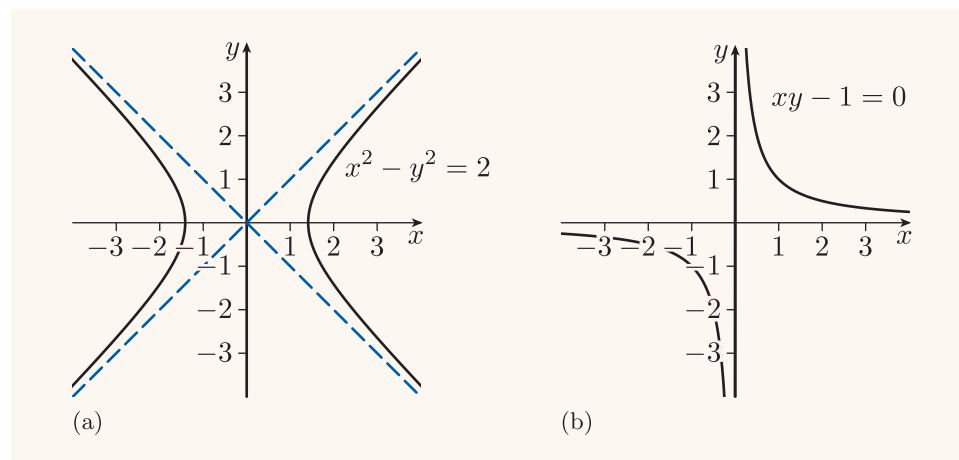


Figure 50 Two congruent hyperbolas

Note that not every equation of the form in the box above is the equation of a conic. For example, the equation $x^2 + y^2 + 1 = 0$ is of the stated form but doesn't represent a conic. To see this, notice that no point (x, y) satisfies this equation, because $x^2 + y^2 \geq 0$ for all possible values of x and y . So this equation doesn't represent any curve at all.

As another example, the equation $x + y + 1 = 0$ is of the stated form but represents a straight line rather than a conic. In general, if all three of the constants A , B and C in the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ are zero (and at least one of D and E is non-zero) then the equation represents a straight line.

If an equation of the form in the box above *does* represent a conic, then you can tell what type of conic it is from the values of the coefficients in the equation, as set out in the following box. The proof of this result is beyond the scope of this module.

Determining the type of a conic

Suppose that the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A , B , C , D , E and F are constants, represents a non-degenerate conic.

- If $B^2 - 4AC < 0$, then the conic is an ellipse or a circle.
- If $B^2 - 4AC = 0$, then the conic is a parabola.
- If $B^2 - 4AC > 0$, then the conic is a hyperbola.

You can distinguish the equation of an ellipse from that of a circle by using the fact that for a circle the coefficient B of the term in xy is zero, and the coefficients A of x^2 and C of y^2 are equal, as you saw in MST124 Unit 5.

Activity 21 *Determining the type of a conic*

Given that the equation

$$3x^2 + 7xy + 4y^2 - 48x - 48y + 324 = 0$$

represents a non-degenerate conic, determine what type of conic it is.

One way to check whether an equation represents a curve, and if so what the curve looks like, is to plot it on a computer. In MST124 you learned how to use the module computer algebra system to plot a curve represented by an equation in implicit form, and this was revised in Section 3 of the MST125 *Computer algebra guide*. In the next activity you're asked to use the computer algebra system to plot some conics.

Activity 22 *Plotting conics on a computer*

The following equations represent non-degenerate conics. Use the computer algebra system to plot them. What types of conic do they seem to be? Check your answers by using the facts in the box above.

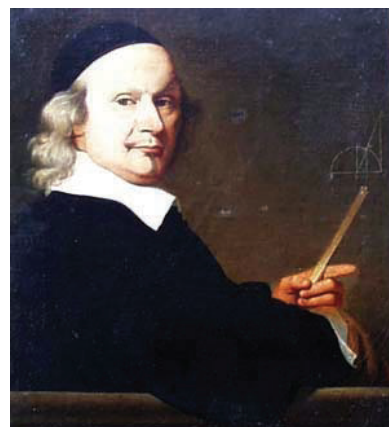
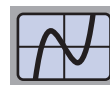
(a) $2x^2 - 3xy + 5y^2 - y - 2 = 0$ (b) $x^2 + 5xy - y^2 + 1 = 0$

Finally in this section, you may be wondering how the definitions of conics as cross-sections of a double cone in Section 1 tie up with the other geometric and algebraic representations of them that you've seen. It's possible to obtain the equation of a conic from its definition as a cross-section of a double cone by following these steps:

- introduce coordinate axes in the slicing plane
- write down the algebraic condition that a point on the slicing plane must satisfy if it lies on the double cone.

Deriving the equations of conics in this way is a fairly complicated process, though it has been known for hundreds of years.

In 1655, the English mathematician John Wallis published a book, *De sectionibus conicis* (*On conic sections*), in which he defined conic sections in algebraic terms and also, incidentally, introduced the symbol ∞ for infinity.



John Wallis (1616–1703)

6 Parametric equations

In this section you'll meet a useful way to describe and work with lines and curves algebraically. It's particularly useful for curves, such as conics in standard position, that don't have equations of the form $y = f(x)$, but it's useful for other lines and curves too, as you'll see.

6.1 What are parametric equations?

You've seen that one way to describe a line or curve is to give an equation that specifies the relationship between the x -coordinate and the y -coordinate of any point on the line or curve. For example, the equation $y = 2x - 3$ describes a particular straight line, and the equation $(x - 2)^2 + (y - 1)^2 = 9$ describes a particular circle.

An alternative way to describe a line or curve is to give a *pair* of equations, which express the x -coordinate and the y -coordinate of any point on the line or curve in terms of a third variable, usually denoted by t . For example, consider the pair of equations

$$x = t + 1 \quad \text{and} \quad y = 2t - 1.$$

Substituting $t = 1$, for example, gives

$$x = 1 + 1 = 2 \quad \text{and} \quad y = 2 \times 1 - 1 = 1.$$

This tells you that $(2, 1)$ is a point on the line or curve described by the pair of equations. Similarly, substituting $t = 0$ tells you that $(1, -1)$ is a point on the line or curve, and so on. Each possible value of t gives a point, and the set of all such points forms the line or curve.

The variable t in a pair of equations like those above is called a **parameter**, and the equations are called **parametric equations**.

We assume that the parameter t in a pair of parametric equations takes all possible real-number values, unless it's stated otherwise. Sometimes, as you'll see, it's necessary or helpful to restrict the parameter t to take values only from a specified range of numbers, which we usually state along with the parametric equations. When this is done, the resulting line or curve may be only part of the line or curve that's obtained when t takes all possible real-number values.

You can describe a line or curve in terms of a parameter t without explicitly stating a pair of parametric equations. For example, you can say that a particular line or curve is given by the set of all points of the form $(t + 1, 2t - 1)$. In general, any way of describing a line or curve in terms of a parameter is called a **parametrisation** of the line or curve. We also say that the line or curve has been **parametrised**.

You can use parametric equations to describe many different kinds of lines and curves. As mentioned above, they're particularly useful for plotting, and working with, curves that can't be described by equations of the form $y = f(x)$. The curves in Figure 51 have all been drawn by a computer

using parametric equations, as you'll see later. In each case, the computer has run the parameter t through a long sequence of values, and used a pair of parametric equations to produce a sequence of corresponding points (x, y) lying on the curve, which it has then plotted.

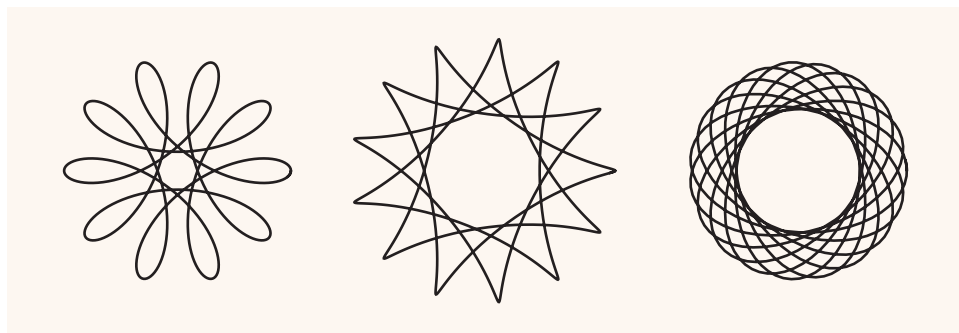


Figure 51 Curves obtained using parametric equations

In this subsection, you'll practise plotting curves from parametric equations, both by hand and using a computer. In the later subsections in this section, you'll learn how to use parametric equations to describe lines, conics and other curves.

Plotting lines or curves given by parametric equations

As you saw in MST124, you can often draw the line or curve given by an equation in x and y by finding some points on it and drawing a smooth line or curve through them. You can use the same approach when the line or curve is given by a pair of parametric equations. To find some points on the line or curve, you can choose some values of the parameter t and substitute them in turn into the parametric equations, as illustrated for a single point in the following example.

Example 6 *Finding a point given by parametric equations*

Find a point on the line or curve given by the parametric equations

$$x = t + 4 \quad \text{and} \quad y = 2t + 3.$$

Solution

Choosing $t = -2$, say, gives

$$x = t + 4 = -2 + 4 = 2 \quad \text{and} \quad y = 2t + 3 = 2 \times (-2) + 3 = -1.$$

So the point $(2, -1)$ lies on the line or curve.

The next activity asks you to calculate the coordinates of some more points given by the parametric equations in Example 6, and to use them to help you plot the line or curve described by the parametric equations.

Activity 23 *Finding points given by parametric equations*

Consider the parametric equations

$$x = t + 4, \quad y = 2t + 3.$$

- (a) Calculate the points corresponding to the parameter values $t = -1, 0, 1$ and 2 .
- (b) Plot these points, and the point from Example 6, and sketch the line or curve through them.

In Activity 23 you should have found that the points all seem to lie on a straight line. You'll see in the next subsection that the parametric equations in the activity do indeed describe a straight line.

In the next activity you should find that the points seem to lie on a curve. Notice also that in this activity the parameter t is restricted to take values only from a specified range of numbers.

Activity 24 *Using parametric equations*

Consider the parametric equations

$$x = 3t^2, \quad y = 6t \quad (-2 \leq t \leq 2).$$

- (a) Calculate the points corresponding to the parameter values $t = -2, -1, 0, 1$ and 2 .
- (b) Plot these points, and hence sketch the line or curve given by the parametric equations and range of values for the parameter t above.

Later in this section you'll see that the curve in Activity 24 is part of a parabola in standard position.

For some pairs of parametric equations, it's impractical to try to plot the corresponding curve by hand, especially if it has an intricate shape. For example, you would need to plot many points to obtain an accurate representation of one of the curves in Figure 51! The next activity shows you how to use the module computer algebra system to plot curves from parametric equations.

Activity 25 Using a computer to plot curves from parametric equations



Work through Subsection 5.1 of the *Computer algebra guide*.

The curves in Figure 51 are examples of curves called *hypotrochoids*. A **hypotrochoid** is a curve that's traced out by a point attached to a circle as it rolls around the inside of another circle, as illustrated in Figure 52. The attached point can be inside or outside the rolling circle, or on its circumference.

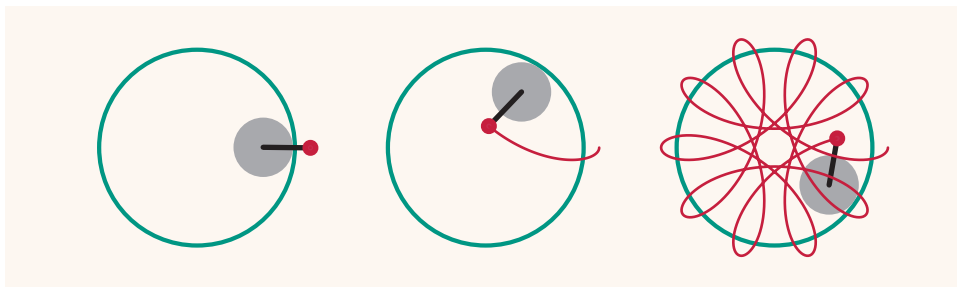


Figure 52 A hypotrochoid being traced out

A hypotrochoid has parametric equations of the form

$$\begin{aligned}x &= (R - r) \cos t + d \cos\left(\frac{R - r}{r} t\right) \\y &= (R - r) \sin t - d \sin\left(\frac{R - r}{r} t\right),\end{aligned}$$

where R and r are the radii of the fixed and rolling circles, respectively, and d is the distance of the point attached to the rolling circle from the centre of that circle, as shown in Figure 53. You don't need to be concerned about understanding how these equations arise, though you might like to think about that. They're presented here just as an interesting example of parametric equations.

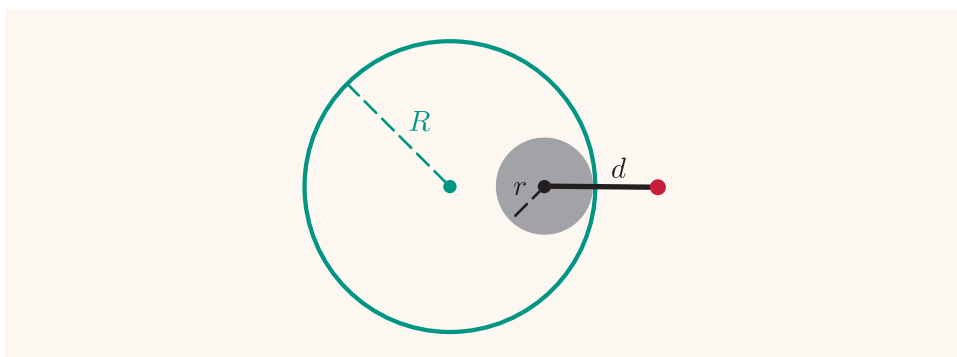


Figure 53 The constants in the parametric equations for a hypotrochoid
In the next activity, you can use an applet to plot some hypotrochoids.



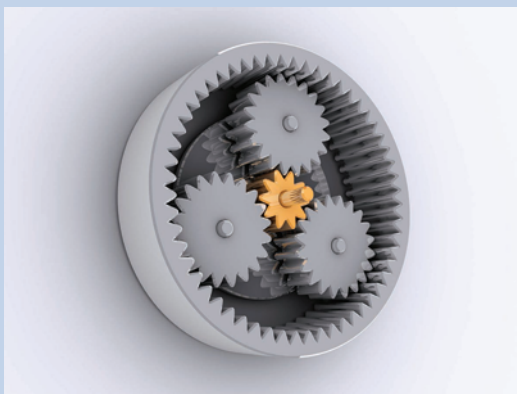
Activity 26 Plotting hypotrochoids

Open the *Hypotrochoids* applet. It shows the hypotrochoid obtained from the parametric equations stated above. You can slowly increase the value of the parameter t to see the curve being traced out. You can also change the radii R and r of the circles, the distance d of the point that traces out the curve from the centre of the rolling circle, and the maximum value of the parameter t .

- Set $R = 1$, $r = 0.3$ and $d = 0.5$. These values give the hypotrochoid in Figure 52, which is also the first hypotrochoid in Figure 51 on page 65. Slowly increase the value of t and observe how the curve is traced out.
- Set $t = 0$, $R = 1$, $r = 0.5$ and $d = 1$. Then slowly increase the value of t . What type of curve appears to be plotted?
- Change the values of R , r and d to explore the different patterns that you can obtain, increasing the value of t to see more of the curves if necessary. Can you obtain patterns of a similar kind to those in Figure 51 on page 65? (Spend only a few minutes on this activity, and note that you have already obtained the first hypotrochoid in Figure 51, in part (a).)

Planetary gears

One practical application of the situation in Figure 52 is in *planetary gear systems*. An example is shown here.



In such a system, a number of small gears run around the inner edge of a larger gear, and mesh with another central gear. There are many advantages of these gear systems, including their compactness and efficiency. They're found in many applications, ranging from power generators to the gearboxes of automatic transmission cars. The curve traced by a point attached to one of the gears touching the outer gear is a hypotrochoid.

6.2 Parametric equations for straight lines

In this subsection you'll learn how to work with, and then how to find, parametric equations for straight lines.

In Activity 23 in the previous subsection you were asked to plot some points given by the parametric equations

$$x = t + 4, \quad y = 2t + 3.$$

You should have found that all the points seem to lie on a straight line, as shown in Figure 54. To show that every point given by these parametric equations lies on the same straight line, no matter what value you take the parameter t to be, you can use the parametric equations to work out an equation that expresses y in terms of x . This process is known as **eliminating** the parameter. It's illustrated in the next example.

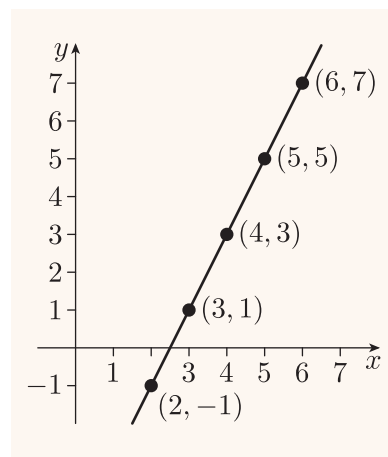


Figure 54 Points given by the parametric equations $x = t + 4$, $y = 2t + 3$

Example 7 Eliminating the parameter from parametric equations

Eliminate the parameter from the parametric equations

$$x = t + 4, \quad y = 2t + 3.$$

Solution

 Express t in terms of x , then substitute into the equation for y . 

The equation $x = t + 4$ gives $t = x - 4$. Hence

$$y = 2(x - 4) + 3 = 2x - 5.$$

So all the points given by the parametric equations lie on the line with equation $y = 2x - 5$.

The solution to Example 7 shows that every point (x, y) given by the parametric equations

$$x = t + 4, \quad y = 2t + 3$$

satisfies the equation $y = 2x - 5$ and hence lies on the line with this equation.

It doesn't show that the parametric equations give *every* point on the line, but in fact they do. To see this, suppose that (x, y) is any point on the line, so the numbers x and y satisfy the equation $y = 2x - 5$. Choose t to be the number that ensures that the first parametric equation $x = t + 4$ is satisfied; that is, choose $t = x - 4$. Then, since x and y satisfy the equation $y = 2x - 5$, we have

$$y = 2x - 5 = 2(t + 4) - 5 = 2t + 3.$$

So the second parametric equation is also satisfied for this value of t . That is, the point (x, y) is given by the parametric equations for the chosen value of t . Since (x, y) represents any point on the line, this shows that every point on the line is given by the parametric equations for some value of t .

This argument, together with the working in Example 7, can be generalised to show that the following facts hold.

Every pair of parametric equations of the form

$$x = at + b, \quad y = ct + d,$$

where a , b , c and d are constants with a and c not both zero, describes a straight line.

To find the equation of the straight line described by a pair of parametric equations of the form in the box above, you just need to eliminate the parameter by using the method demonstrated in Example 7. You're asked to do that in the next activity.

Activity 27 *Eliminating the parameter from parametric equations*

Find the equation of the line described by the parametric equations

$$x = 3t + 1, \quad y = -12t + 5.$$

If, instead of allowing the parameter t in a pair of parametric equations for a straight line to take all real-number values, you restrict the values taken by t to some interval, then the parametric equations describe a part of the line, rather than the whole line. This is illustrated in the next example, which uses the parametric equations from Example 7 again, but with the values of t restricted.

Example 8 *Restricting the values taken by the parameter in parametric equations for a straight line*

Draw the graph given by the following pair of parametric equations and range of values for the parameter t :

$$x = t + 4, \quad y = 2t + 3 \quad (-2 \leq t < 0).$$

Solution

The parametric equations are of the form $x = at + b$, $y = ct + d$, so with the given range of values for t they describe part of a straight line.

 Find the points given by the endpoints of the interval, and check whether each endpoint is included or not. 

Taking $t = -2$ gives

$$x = -2 + 4 = 2 \quad \text{and} \quad y = 2(-2) + 3 = -1,$$

so an endpoint of the graph is $(2, -1)$. This endpoint is included.

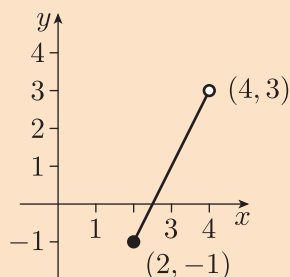
Taking $t = 0$ gives

$$x = 0 + 4 = 4 \quad \text{and} \quad y = 2 \times 0 + 3 = 3,$$

so the other endpoint is $(4, 3)$. This endpoint is excluded.

So the parametric equations describe the part of the line from $(2, -1)$ up to but not including $(4, 3)$.

The graph is shown here.



Remember that, in graphs like the one in Example 8, a solid dot indicates an endpoint that is included and a hollow dot indicates one that isn't.

Activity 28 *Restricting the values taken by the parameter in parametric equations for a straight line*

Draw the graph given by the following pair of parametric equations and range of values for the parameter t :

$$x = 1 - 2t, \quad y = t + 1 \quad (-1 < t \leq 3).$$

In the next activity you can explore how the point on a straight line given by a particular value of the parameter t moves along the line as you change the value of t .


Activity 29 Exploring parametric equations for straight lines

Open the *Parametric equations* applet.

- (a) Check that the parametric equations and the minimum and maximum values of the parameter t are set to those in Example 8, namely

$$x = t + 4, \quad y = 2t + 3 \quad (-2 \leq t < 0).$$

- (i) Gradually increase the value of t from -2 to 0 , and observe how the corresponding point P traces out the line segment.
 - (ii) By setting t equal to -2 and then to 0 , check that the endpoints of the line segment are $(2, -1)$ and $(4, 3)$, as found in Example 8.
 - (iii) What value of t do you think corresponds to the midpoint of the line segment? Use the applet to confirm your answer (at least approximately).
- (b) Now change the parametric equations and the minimum and maximum values of t to those from Activity 28, as follows:

$$x = 1 - 2t, \quad y = t + 1 \quad (-1 < t \leq 3).$$

- (i) Gradually increase the value of t from -1 to 3 , and observe how the corresponding point P traces out the line segment.
- (ii) By setting t equal to -1 and then to 3 , check that the endpoints of the line segment are $(3, 0)$ and $(-5, 4)$, as found in the solution to Activity 28.
- (iii) What value of t do you think corresponds to the midpoint of the line segment? What value of t do you think corresponds to the point that's a quarter of the way along the line segment from the endpoint corresponding to $t = -1$ to the endpoint corresponding to $t = 3$? Use the applet to confirm your answers (at least approximately).

When you're working with parametric equations, it can sometimes be helpful to think of the parameter t as representing time. Then the point P given by the parametric equations moves along the line or curve as time moves on; that is, as t increases.

This way of thinking can help you see the reasons behind the results that you should have obtained in parts (a)(iii) and (b)(iii) of Activity 29. Consider, for example, the parametric equations from part (a), which are

$$x = t + 4, \quad y = 2t + 3.$$

The coefficient of t in the x -coordinate $t + 4$ of the point $P(x, y)$ is 1, so whenever t increases by 1, the x -coordinate of P increases by 1. Similarly, the coefficient of t in the y -coordinate $2t + 3$ of P is 2, so whenever t increases by 1, the y -coordinate of P increases by 2. Altogether, whenever t increases by 1, the point P moves 1 unit right and 2 units up. So the point P moves along the line at a *constant* speed as t increases. Hence, for

example, in a quarter of the ‘time’ interval from $t = -2$ to $t = 0$, the point P moves a quarter of the distance that it moves in the whole time interval.

You can see that for any parametric equations of the form

$$x = at + b, \quad y = ct + d,$$

where a , b , c and d are constants with a and c not both zero, the point P moves along the straight line given by the parametric equations at a constant speed as t increases. This isn’t the case for all parametrisations, however.

Taking a parameter t to represent time can also help you to solve some types of practical problems. When a problem involves the motion of an object, it can be helpful to specify the coordinates of the position of the object in terms of the time. If time is represented by the variable t , then the equations for the x - and y -coordinates in terms of t form a pair of parametric equations with the parameter t . You can calculate the position of the object at any particular time by substituting the appropriate value of t into the equations.

The next activity demonstrates this approach, by using an applet to illustrate the motion of two ships, each moving along a straight-line path.

Activity 30 Using time as the parameter



Open the *Closest approach* applet. Check that the parametric equations are set as follows:

point A : $x = 2t - 3, \quad y = -t + 13$

point B : $x = 3t - 10, \quad y = t + 4.$

Check also that the minimum and maximum values of the parameter t are set to 0 and 10, respectively.

The idea here is that the points A and B represent two ships, ship A and ship B . At any time t (in hours) with $0 \leq t \leq 10$, the position of ship A is given by $(2t - 3, -t + 13)$, and the position of ship B is given by $(3t - 10, t + 4)$. These coordinates are with reference to some chosen coordinate system, with distance measured in kilometres.

- Gradually increase the value of t from 0 to 10, and observe the paths traced out by the two ships. Do the ships collide?
- The applet displays the distance between the ships at the currently set value of t . By changing the value of t and using this display, find (approximately) the closest distance to which the ships approach each other, and the time at which this closest distance occurs.

The parametric equations in Activity 30 are as follows:

$$\text{point A: } x = 2t - 3, \quad y = -t + 13$$

$$\text{point B: } x = 3t - 10, \quad y = t + 4.$$

The coefficients of the parameter t in these equations tell you the following. In each hour that passes, ship A travels 2 km right and 1 km down, with reference to the chosen coordinate system. In other words, it has velocity 2 km h^{-1} in the x -direction and velocity -1 km h^{-1} in the y -direction. Similarly, ship B has velocity 3 km h^{-1} in the x -direction and velocity 1 km h^{-1} in the y -direction. If you wish, you can work out the overall velocity of either ship by using the ideas about vectors that are covered in MST124 and revised in MST125 Unit 1.

The next example illustrates how to find an exact answer for the closest approach of the two ships in Activity 30, by working by hand.



Example 9 Finding the closest approach of two ships

Two ships pursue straight-line courses at steady speeds. At each time t (in hours) with $0 \leq t \leq 10$, the position of ship A is given by $(2t - 3, -t + 13)$, and the position of ship B is given by $(3t - 10, t + 4)$. These coordinates are with reference to a particular coordinate system, with distance measured in kilometres.

What is the closest distance to which the ships approach each other, and when does this closest distance occur?

Solution

Find an expression for the distance between the two ships in terms of the time t . Use the distance formula, which says that the distance between the two points (x_1, y_1) and (x_2, y_2) is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

At time t (in hours), ship A is at $(2t - 3, -t + 13)$ and ship B is at $(3t - 10, t + 4)$. So the distance d (in kilometres) between them is given by

$$d^2 = (3t - 10 - (2t - 3))^2 + (t + 4 - (-t + 13))^2$$

Simplify this expression for d^2 .

$$\begin{aligned} &= (t - 7)^2 + (2t - 9)^2 \\ &= t^2 - 14t + 49 + 4t^2 - 36t + 81 \\ &= 5t^2 - 50t + 130. \end{aligned}$$

The closest approach occurs when d , and hence d^2 , takes its minimum value.

☁ The expression for d^2 is a quadratic, so use the technique of completing the square. ☁

Completing the square gives

$$\begin{aligned} d^2 &= 5(t^2 - 10t) + 130 \\ &= 5((t - 5)^2 - 25) + 130 \\ &= 5(t - 5)^2 - 125 + 130 \\ &= 5(t - 5)^2 + 5. \end{aligned}$$

☁ Since the value of the squared term is always positive or zero, the minimum value of d^2 occurs when the squared term is zero. ☁

The term $5(t - 5)^2$ never takes a value less than 0, and it takes the value 0 when $t - 5 = 0$; that is, when $t = 5$. At this value of t the value of d^2 is 5, which gives $d = \sqrt{5} = 2.24$ (to 2 d.p.).

Hence the closest distance is approximately 2.24 km, and it occurs 5 hours after the start time.

Here's a similar example for you to try. You might like to check your answers by using the *Closest approach* applet. You can change the parametric equations and the range of values for the parameter t in the applet.

Activity 31 Finding the closest approach of two aircraft

Two aircraft fly along straight-line courses at the same height and at steady speeds. At each time t (in hours) with $0 \leq t \leq 8$, the position of aircraft A is given by $(496t + 22, -310t - 23)$, and the position of aircraft B is given by $(500t + 10, -312t - 12)$. These coordinates are with reference to a particular coordinate system, with distance measured in kilometres.

What is the closest distance to which the aircraft approach each other, and when does this closest distance occur?

Now that you've seen some uses of parametric equations for straight lines, in the rest of this subsection you'll learn how to *obtain* parametric equations for straight lines.

Let's start by considering the example of the straight line with the equation $y = 2x + 3$, which has gradient 2 and y -intercept 3. The equation of the line tells you that the y -coordinate of each point on the line is equal to twice the x -coordinate of that point, plus 3. Another way to specify this is to say that the line consists of all points of the form $(x, 2x + 3)$, where x is any real number.

You can equally well use another letter in place of x here, such as t . So you can say that the line consists of all points of the form $(t, 2t + 3)$, where t is any real number. In other words, parametric equations for the line are

$$x = t, \quad y = 2t + 3.$$

However, there are infinitely many other possible parametrisations of this line. For example, if you return to the coordinates $(x, 2x + 3)$, and replace x by $4t + 1$ instead of t , then you obtain the following parametrisation:

$$x = 4t + 1, \quad y = 2(4t + 1) + 3 = 8t + 5.$$

You might like to try coming up with another parametrisation for the same line. You can check it by eliminating the parameter.

In general, any straight line has infinitely many parametrisations. However, there are two that are particularly useful, which we'll consider next.

Obtaining a parametrisation of a straight line from a point on it and its gradient

Sometimes it's convenient to use a point on a non-vertical straight line and the gradient of the line to find a parametrisation that relates directly to these two features.

Consider, for example, the straight line that passes through the point $(5, -2)$ and has gradient 3, as shown in Figure 56.

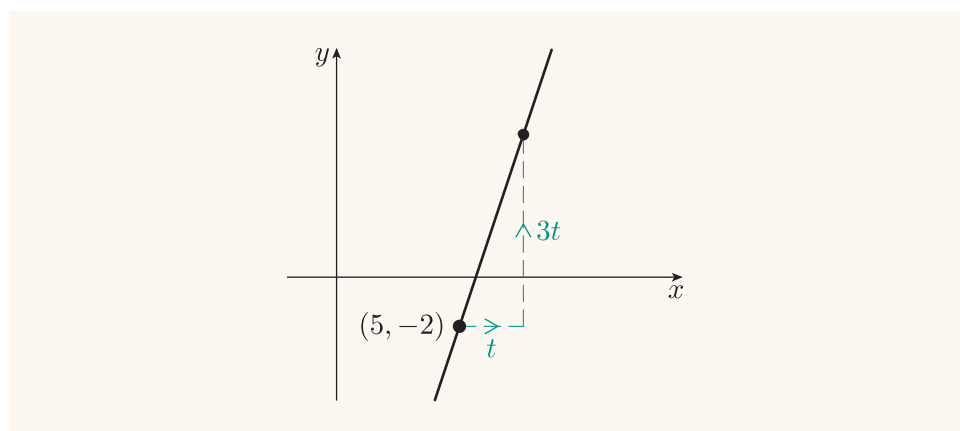


Figure 55 The line that passes through the point $(5, -2)$ and has gradient 3

Since the line has gradient 3, you can reach any point on it by placing your pen tip at the point $(5, -2)$ and moving it t units right and $3t$ units up, for some number t . The value of t can be positive, negative or zero.

(Remember that if you move your pen tip a *negative* number of units right then you actually move it left, and similarly if you move it a *negative* number of units up then you actually move it down.)

So each point on the line is of the form

$$(5 + t, -2 + 3t),$$

for some value of t . In other words, the following equations are parametric equations for the line:

$$x = 5 + t, \quad y = -2 + 3t.$$

Notice that taking $t = 0$ in these equations gives $(x, y) = (5, -2)$, so this parametrisation corresponds to thinking of the given point on the line, $(5, -2)$, as the ‘starting point’.

In general, consider the straight line that passes through the point (x_0, y_0) and has gradient m , as shown in Figure 56.

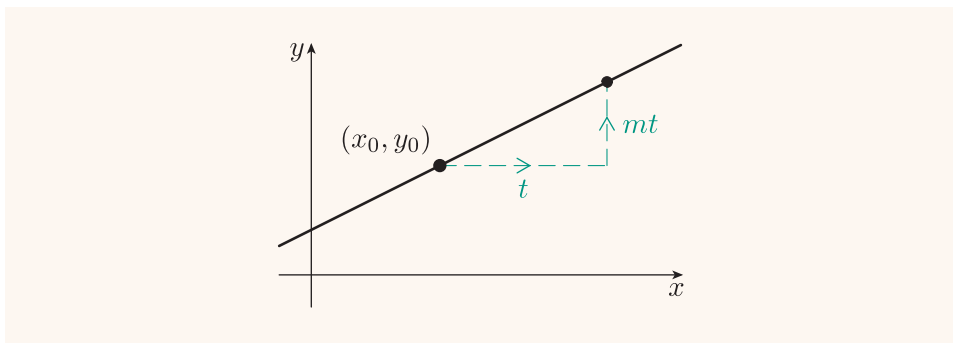


Figure 56 The line that passes through the point (x_0, y_0) and has gradient m

You can reach any point on the line by placing your pen tip at the point (x_0, y_0) and moving it t units to the right and mt units up, for some number t . So parametric equations for the line are

$$x = x_0 + t, \quad y = y_0 + mt.$$

Taking $t = 0$ gives $(x, y) = (x_0, y_0)$, so this parametrisation corresponds to thinking of the point (x_0, y_0) as the ‘starting point’. This type of parametrisation of a straight line is summarised as follows.

Parametric equations for a straight line, arising from a point on it and its gradient

The line that passes through the point (x_0, y_0) and has gradient m has parametric equations

$$x = x_0 + t, \quad y = y_0 + mt.$$

With this parametrisation, $t = 0$ corresponds to the point (x_0, y_0) .

Example 10 Finding a parametrisation of a straight line from a point on it and its gradient

Write down parametric equations for the straight line that passes through the point $(-3, 1)$ and has gradient -2 .

Solution

Take $(x_0, y_0) = (-3, 1)$ and $m = -2$ in the box above.

Suitable parametric equations are

$$x = -3 + t, \quad y = 1 - 2t.$$

Activity 32 Finding a parametrisation of a straight line from a point on it and its gradient

Write down parametric equations for the straight line that passes through the point $(-2, 3)$ and has gradient 4 .

Obtaining a parametrisation of a straight line from two points on it

Sometimes it's useful to use two points on a straight line to obtain a parametrisation that relates directly to those two points.

Consider the straight line that passes through the points (x_0, y_0) and (x_1, y_1) . The run and the rise from the first point to the second point are $x_1 - x_0$ and $y_1 - y_0$, respectively, as illustrated in Figure 57(a). So if you trace your pen tip along the line from left to right (this assumes that the line is not vertical), then for every $x_1 - x_0$ units that it moves to the right, it moves $y_1 - y_0$ units up.

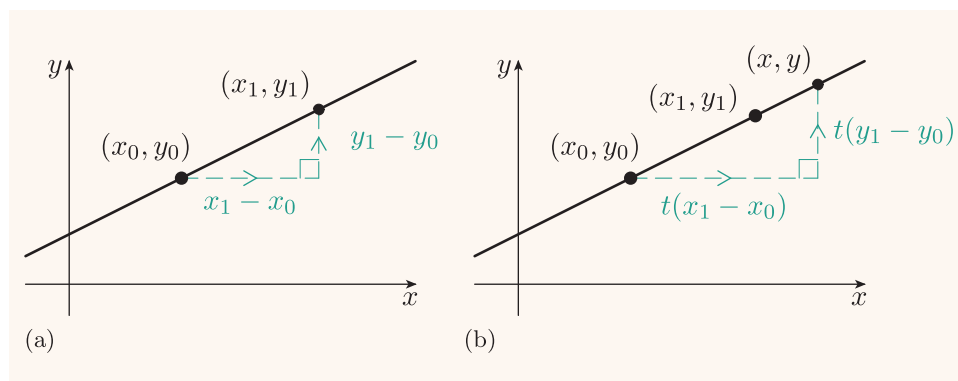


Figure 57 The line through the points (x_0, y_0) and (x_1, y_1)

Hence you can reach any point (x, y) on the line by placing your pen tip at the point (x_0, y_0) and moving it $t(x_1 - x_0)$ units to the right and $t(y_1 - y_0)$ units up, for some number t , as illustrated in Figure 57(b). In fact, this last statement holds even if the line is vertical: in this case you move your pen tip by $t(x_1 - x_0) = 0$ units to the right and $t(y_1 - y_0)$ units up, for some number t .

It follows that parametric equations for the line are

$$x = x_0 + t(x_1 - x_0) \quad \text{and} \quad y = y_0 + t(y_1 - y_0).$$

Note that taking $t = 0$ in these equations gives

$$x = x_0 \quad \text{and} \quad y = y_0,$$

while taking $t = 1$ gives

$$x = x_0 + (x_1 - x_0) = x_1 \quad \text{and} \quad y = y_0 + (y_1 - y_0) = y_1.$$

So the parameter values $t = 0$ and $t = 1$ give the original points (x_0, y_0) and (x_1, y_1) , respectively. In particular, as with the parametrisation of a straight line from a point on it and its gradient that you met earlier, with this parametrisation you can think of the point (x_0, y_0) as the ‘starting point’.

This type of parametrisation of a straight line is summarised as follows.

Parametric equations for a straight line, arising from two points on it

The line that passes through the points (x_0, y_0) and (x_1, y_1) has parametric equations

$$x = x_0 + t(x_1 - x_0), \quad y = y_0 + t(y_1 - y_0).$$

With this parametrisation, $t = 0$ corresponds to the point (x_0, y_0) and $t = 1$ corresponds to the point (x_1, y_1) .

You can use the fact in the box above in the next activity. Remember that to specify a parametrisation of a line or curve you must give not only suitable parametric equations, but also an appropriate range of values for the parameter t , unless t can be any real number.

Activity 33 Finding parametrisations for straight lines from two points on them

Write down a parametrisation of each of the following lines or line segments.

- The line that passes through the two points $(1, 5)$ and $(4, -7)$.
- The line segment that joins the points $(-2, 4)$ and $(3, 1)$.

6.3 Parametric equations for circles

Now let's see how to find parametric equations for circles.

First, consider the unit circle; that is, the circle whose centre is the origin O and whose radius is 1, as shown in Figure 58. By the definitions of the sine and cosine functions, if $P(x, y)$ is any point on this circle, and t is the angle in radians measured anticlockwise from the x -axis to the line OP , then

$$x = \cos t \quad \text{and} \quad y = \sin t.$$

These equations are parametric equations for the unit circle. You're probably more familiar with them with the angle denoted by θ rather than t , but we're using t here because in this unit we usually denote the parameter in parametric equations by t .

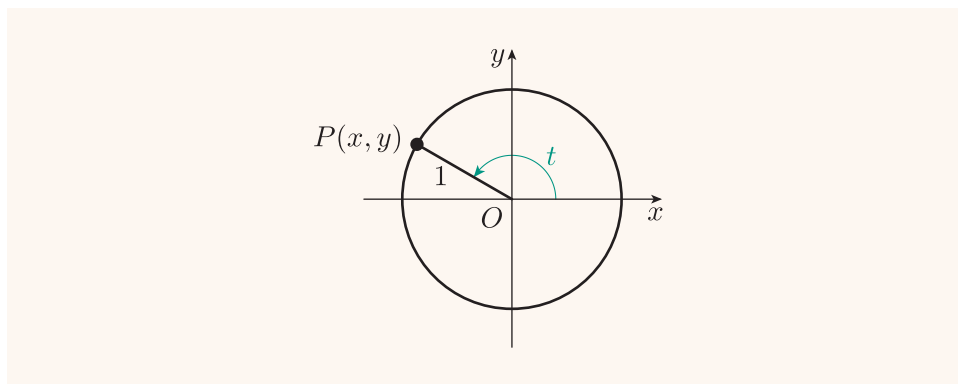


Figure 58 The unit circle

Figure 59 shows several values of the parameter t , and the corresponding points given by the parametric equations above. For example, when $t = \pi/2$,

$$x = \cos \frac{\pi}{2} = 0 \quad \text{and} \quad y = \sin \frac{\pi}{2} = 1.$$

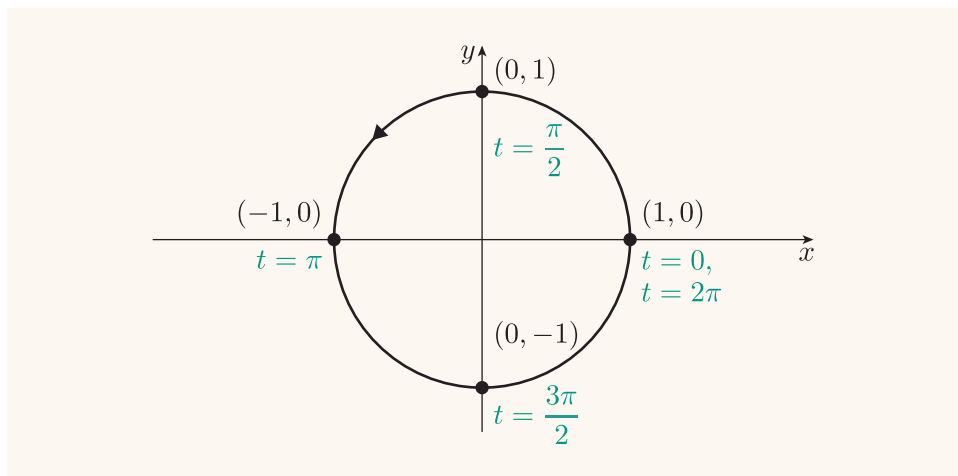


Figure 59 Points on the unit circle

The arrow on the circle shows the direction of motion of the point (x, y) around the circle as t increases from 0 to 2π .

The parametrisation

$$x = \cos t, \quad y = \sin t,$$

of the unit circle differs in an important way from the parametrisations of straight lines that you've seen. Once t has taken all the values from 0 to 2π , say, the points given by the parametric equations start repeating, since

$$\cos(t + 2\pi) = \cos t \quad \text{and} \quad \sin(t + 2\pi) = \sin t.$$

So each point on the circle corresponds to infinitely many values of t .

You can avoid this situation by restricting the values of t . For example, the parametrisation

$$x = \cos t, \quad y = \sin t \quad (0 \leq t \leq 2\pi)$$

represents a single revolution around the unit circle, starting and finishing at the point $(1, 0)$. Note that even though $t = 0$ and $t = 2\pi$ give the same point, it's conventional to include both values in the range for t .

If you want to describe a part of the unit circle, rather than the whole circle, then you can specify an even smaller range of values for the parameter t . For example, the parametrisation

$$x = \cos t, \quad y = \sin t \quad (0 \leq t \leq \pi)$$

describes the semicircle in Figure 60(a). Similarly, the parametrisation

$$x = \cos t, \quad y = \sin t \quad (\pi \leq t \leq 2\pi)$$

describes the semicircle in Figure 60(b). An alternative parametrisation for the semicircle in Figure 60(b) is

$$x = \cos t, \quad y = \sin t \quad (-\pi \leq t \leq 0).$$

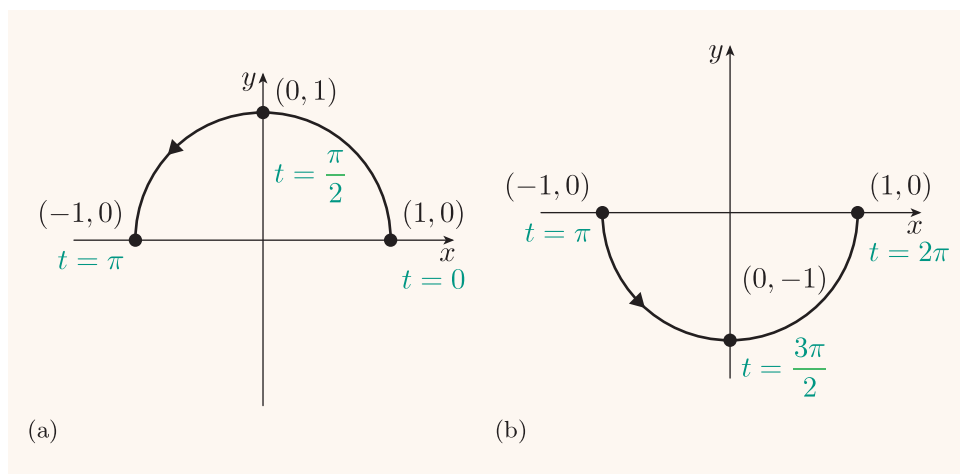


Figure 60 Two semicircles

Now let's consider a circle whose centre is the origin and whose radius is any positive number r , as shown in Figure 61.

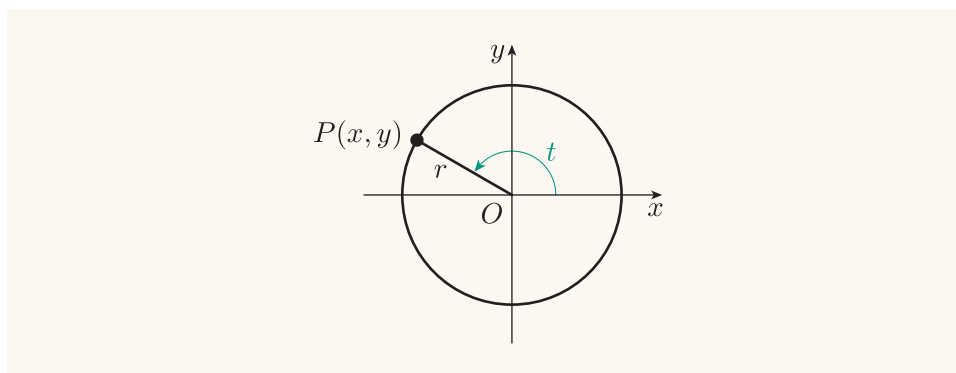


Figure 61 The circle with centre O and radius r

Figure 61 is the same as Figure 58, except that the radius of the circle is scaled by the factor r . It follows that the x - and y -coordinates of the point P are also scaled by the factor r . So parametric equations for the circle in Figure 61 are

$$x = r \cos t, \quad y = r \sin t.$$

For example, the circle whose centre is the origin and whose radius is 3 has parametric equations

$$x = 3 \cos t, \quad y = 3 \sin t.$$

It's now straightforward to find parametric equations for any circle, with any centre and any radius. Consider the circle whose centre is the point (p, q) and whose radius is r . This circle can be obtained from the circle whose centre is the origin $(0, 0)$ and whose radius is r by translating it p units right and q units up, as shown in Figure 62. As usual, the values of p and q may be positive, negative or zero.

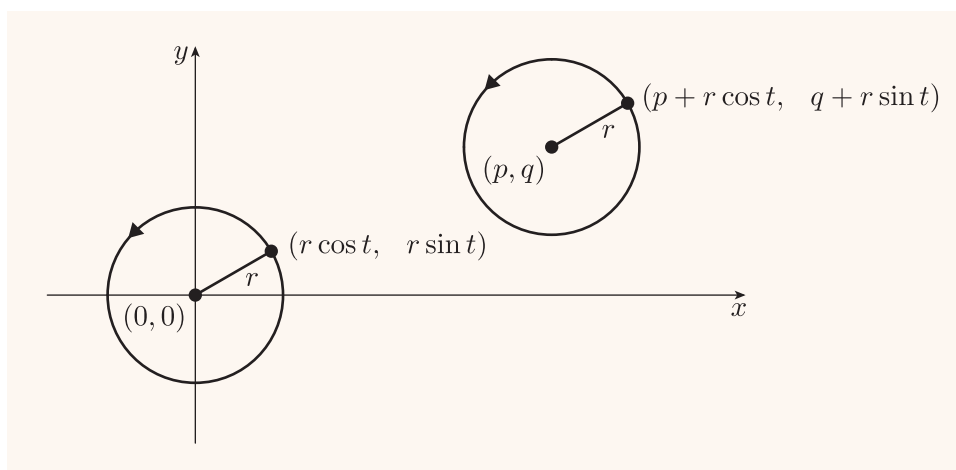


Figure 62 Two circles with radius r

Consider any point on the circle with centre $(0, 0)$ and radius r . It has coordinates (x, y) given by

$$x = r \cos t, \quad y = r \sin t$$

for some value of t . When the circle is translated by p units to the right and q units up, this point is translated to the point with coordinates (x, y) given by

$$x = p + r \cos t, \quad y = q + r \sin t,$$

as shown in Figure 62. So these equations are parametric equations for the translated circle.

For example, the circle whose centre is the point $(5, -1)$ and whose radius is 3 has the parametric equations

$$x = 5 + 3 \cos t, \quad y = -1 + 3 \sin t.$$

Here's a summary of this type of parametrisation of a circle.

Parametric equations for a circle

The circle with centre (p, q) and radius r has parametric equations

$$x = p + r \cos t, \quad y = q + r \sin t.$$

As with straight lines, there are many other possible parametrisations of circles. For example, an alternative parametrisation of the circle with centre $(5, -1)$ and radius 3, mentioned above, is

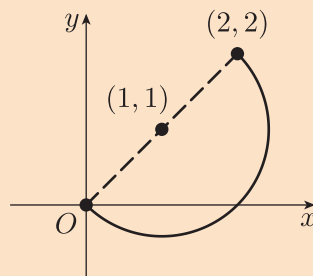
$$x = 5 + 3 \cos(2t), \quad y = -1 + 3 \sin(2t).$$

With this parametrisation, the point (x, y) travels twice as fast round the circle as t increases, compared to the first parametrisation given above.

As with the unit circle, you can use parametric equations to describe part of a circle by restricting the range of values taken by the parameter t . This is illustrated in the next example.


Example 11 *Finding a parametrisation of a semicircle*

Write down a parametrisation for the semicircle with centre $(1, 1)$ shown here.


Solution

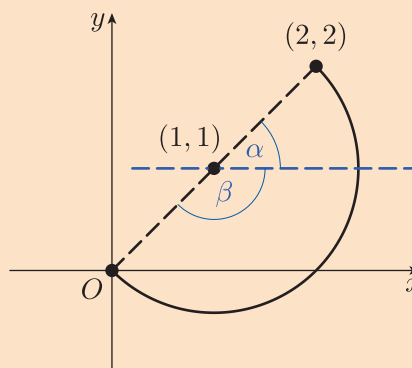
Write down the centre and radius of the semicircle, and hence write down parametric equations for a circle with this centre and radius.

The centre of the semicircle is $(1, 1)$ and the radius is $\sqrt{1^2 + 1^2} = \sqrt{2}$.

Parametric equations for the circle with this centre and radius are

$$x = 1 + \sqrt{2} \cos t, \quad y = 1 + \sqrt{2} \sin t.$$

Work out a suitable range of values for the parameter t , using a diagram to help.



The angle α in the diagram is the angle of inclination of the line through the points $(1, 1)$ and $(2, 2)$, which is $\pi/4$. Hence the angle β in the diagram is given by

$$\beta = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Hence a suitable range of values for t is $-\frac{3\pi}{4} \leq t \leq \frac{\pi}{4}$.

A parametrisation of the semicircle is therefore

$$x = 1 + \sqrt{2} \cos t, \quad y = 1 + \sqrt{2} \sin t \quad \left(-\frac{3\pi}{4} \leq t \leq \frac{\pi}{4} \right).$$

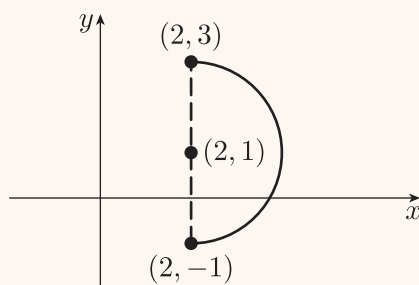
An alternative suitable range of values for the parameter t in Example 11 is $\frac{5\pi}{4} \leq t \leq \frac{9\pi}{4}$.

Here's a similar example for you to try.

Activity 34 Finding parametrisations for circles and semicircles

Write down a parametrisation for each of the following curves.

- The circle with centre $(-3, 1)$ and radius 4.
- The semicircle shown below.



In the next activity you're asked to use the module computer algebra system to plot lines, circles and circular arcs by using parametric equations. A **circular arc** is an unbroken part of a circle.

Activity 35 Using the CAS to plot lines, circles and circular arcs



Use the module computer algebra system and the parametrisations that you found in Activity 33 and Activity 34 to plot the following line segment, circle and semicircle.

- The line segment joining the points $(-2, 4)$ and $(3, 1)$.
- The circle with centre $(-3, 1)$ and radius 4.
- The semicircle in Activity 34(b).

6.4 Parametric equations for conics in standard position

In this subsection you'll meet the standard parametrisations of parabolas, ellipses and hyperbolas in standard position. As with all parametrisations, there are many other possible parametrisations of these curves.

Parabolas

You've seen that a parabola in standard position, as illustrated in Figure 63, has the equation

$$y^2 = 4ax, \quad \text{where } a > 0.$$

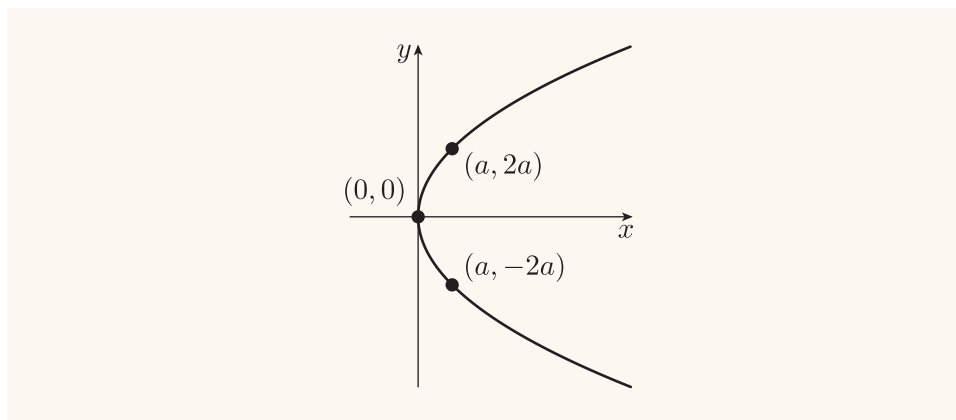


Figure 63 The parabola in standard position with equation $y^2 = 4ax$

For each value of y , this equation gives exactly one corresponding value of x . So we can parametrise the parabola by using the y -coordinate as the parameter; that is, by taking $y = t$. Substituting into the equation of the parabola then gives $x = t^2/(4a)$. So a parametrisation of the parabola is

$$x = \frac{t^2}{4a}, \quad y = t.$$

However, there's a more convenient parametrisation, which avoids the fraction here. We can instead take $y = 2at$, which gives

$$x = \frac{y^2}{4a} = \frac{(2at)^2}{4a} = at^2.$$

This is the standard parametrisation of a parabola in standard position.

Standard parametrisation of a parabola in standard position

The parabola with equation $y^2 = 4ax$ where $a > 0$ has the parametrisation

$$x = at^2, \quad y = 2at.$$

With this parametrisation, the three parameter values $t = -1, 0$ and 1 correspond to the points $(a, -2a)$, $(0, 0)$ and $(a, 2a)$, respectively, which, as you saw earlier, are useful points to plot when you want to sketch the parabola. Figure 64 shows several values of the parameter t , and the corresponding points on the parabola in standard position given by the parametric equations above. The arrow shows the direction of motion of the point (x, y) along the parabola as the value of the parameter t increases.

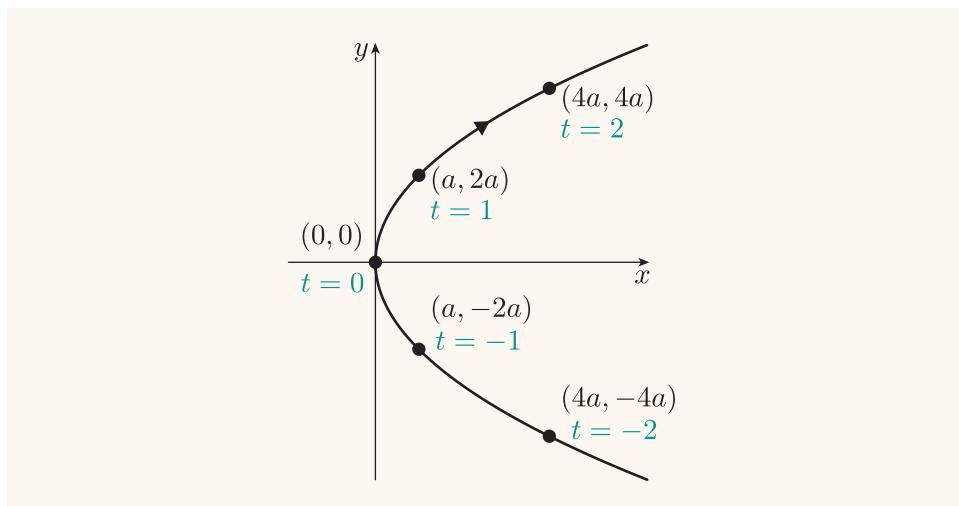


Figure 64 Points on a parabola in standard position

Notice that, with this parametrisation, positive values of t give points on the upper half of the parabola, and negative values of t give points on the lower half.

Activity 36 *Parametrising a parabola in standard position*

- Write down the standard parametrisation of the parabola with equation $y^2 = 2x$.
- Calculate the points corresponding to the parameter values $t = -1, 0, 1, 2$ and 3 .
- Plot these points, and hence draw the parabola, showing the direction of motion of the point (x, y) as t increases.

In Activity 24 on page 66 you were asked to plot some points on the curve with parametrisation

$$x = 3t^2, \quad y = 6t \quad (-2 \leq t \leq 2),$$

and hence sketch the curve. You can now see that this parametrisation is of the form

$$x = at^2, \quad y = 2at \quad (-2 \leq t \leq 2),$$

with $a = 3$, so the curve is part of a parabola in standard position.

Ellipses

You've seen that the ellipse in standard position with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } a > b > 0),$$

is obtained by scaling the circle with equation $x^2 + y^2 = a^2$ vertically by the factor b/a .

You've also seen that the circle with equation $x^2 + y^2 = a^2$ has the parametrisation

$$x = a \cos t, \quad y = a \sin t.$$

If you scale this circle vertically by the factor b/a , then each point $(a \cos t, a \sin t)$ on it moves to the point with the same x -coordinate, namely $a \cos t$, and y -coordinate $(b/a)a \sin t = b \sin t$, as shown in Figure 65.

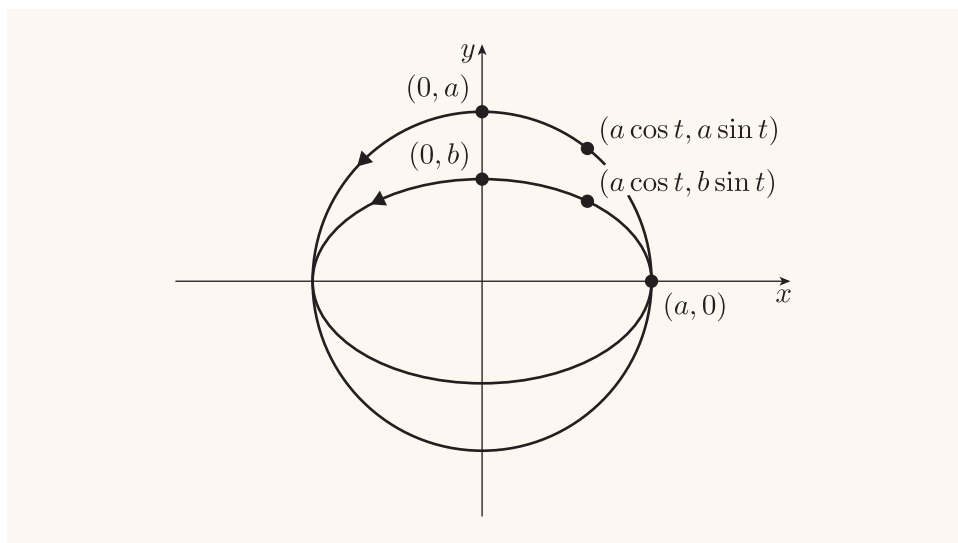


Figure 65 Corresponding points on an ellipse and circle

This gives the following parametrisation of an ellipse in standard position.

Standard parametrisation of an ellipse in standard position

The ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } a > b > 0)$$

has the parametrisation

$$x = a \cos t, \quad y = b \sin t \quad (0 \leq t \leq 2\pi).$$

As t increases from 0 to 2π , the point $(a \cos t, b \sin t)$ travels once round the ellipse in the direction of the arrow in Figure 66, starting and finishing at $(a, 0)$.

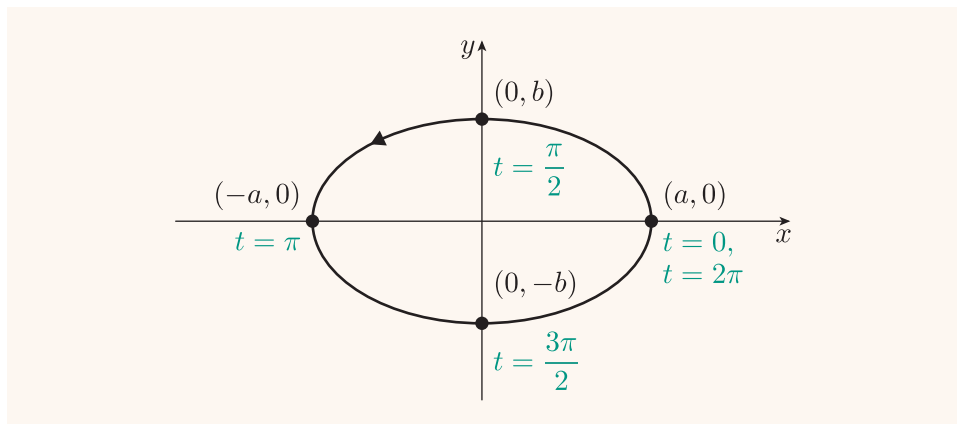


Figure 66 Points on an ellipse in standard position

Activity 37 Parametrising an ellipse

Write down the standard parametrisation of the ellipse with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

Activity 38 Showing that a hypotrochoid can be an ellipse

Earlier in this section you saw that a hypotrochoid has parametric equations of the form

$$\begin{aligned} x &= (R - r) \cos t + d \cos\left(\frac{R - r}{r} t\right) \\ y &= (R - r) \sin t - d \sin\left(\frac{R - r}{r} t\right). \end{aligned}$$

You saw in Activity 26 that if you take $R = 1$, $r = 0.5$ and $d = 1$ in these equations then the curve that you obtain seems to be an ellipse. In this activity you'll see why.

- (a) Show that, for the values of R , r and d given above, the parametric equations reduce to

$$x = 1.5 \cos t, \quad y = -0.5 \sin t.$$

- (b) Explain why these parametric equations give an ellipse.

Hyperbolas

Since the equation of an hyperbola in standard position,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (\text{where } a, b > 0),$$

closely resembles the equation of an ellipse in standard position, you might expect the two types of curve to have similar parametric equations. In fact they do, because you can parametrise the hyperbola with the equation above by using the equations

$$x = a \sec t, \quad y = b \tan t. \quad (9)$$

Remember that $\sec t = 1/\cos t$ and $\tan t = \sin t/\cos t$.

To check that equations (9) give points on the hyperbola, you can show that each point given by these equations satisfies the equation of the hyperbola. Using equations (9) to substitute in the left-hand side of the equation of the hyperbola gives

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{a^2 \sec^2 t}{a^2} - \frac{b^2 \tan^2 t}{b^2} = \sec^2 t - \tan^2 t = 1,$$

where the final step here follows from the trigonometric identity $\tan^2 \theta + 1 = \sec^2 \theta$. This shows that the points given by the parametric equations do satisfy the equation of the hyperbola. So every point given by the parametric equations lies on the hyperbola.

To check whether the parametric equations give *all* the points on the hyperbola, and to find a suitable range of values for the parameter t , we need to think about how the point (x, y) moves along the hyperbola as the parameter t changes.

First, remember that if $t = -\pi/2$, $t = \pi/2$ or $t = 3\pi/2$, and in general if t is equal to $\pi/2$ plus an integer multiple of π , then $\cos t = 0$, and hence $\sec t$ and $\tan t$ are not defined. So for these values of t the parametric equations are not valid. If t is close to one of these values, then $\cos t$ is close to 0, so $\sec t$ and $\tan t$ are both large in magnitude and hence the corresponding point $(x, y) = (a \sec t, b \tan t)$ is far from the origin.

Also, taking $t = 0$ gives

$$(x, y) = (a \sec 0, b \tan 0) = (a \times 1, b \times 0) = (a, 0),$$

and taking $t = \pi$ gives

$$(x, y) = (a \sec \pi, b \tan \pi) = (a \times (-1), b \times 0) = (-a, 0).$$

So $t = 0$ and $t = \pi$ give the points where the hyperbola crosses the x -axis.

In fact, as t increases from $-\pi/2$ to $\pi/2$ (but excluding these values), the point $(x, y) = (a \sec t, b \tan t)$ moves upwards along the entire right-hand branch of the hyperbola, crossing the x -axis when $t = 0$. Similarly, as t increases from $\pi/2$ to $3\pi/2$ (but excluding these values), the point $(\sec t, \tan t)$ moves upwards along the entire left-hand branch of the hyperbola, crossing the x -axis when $t = \pi$. This is illustrated in Figure 67. Other values of t just repeat the same points on the hyperbola.

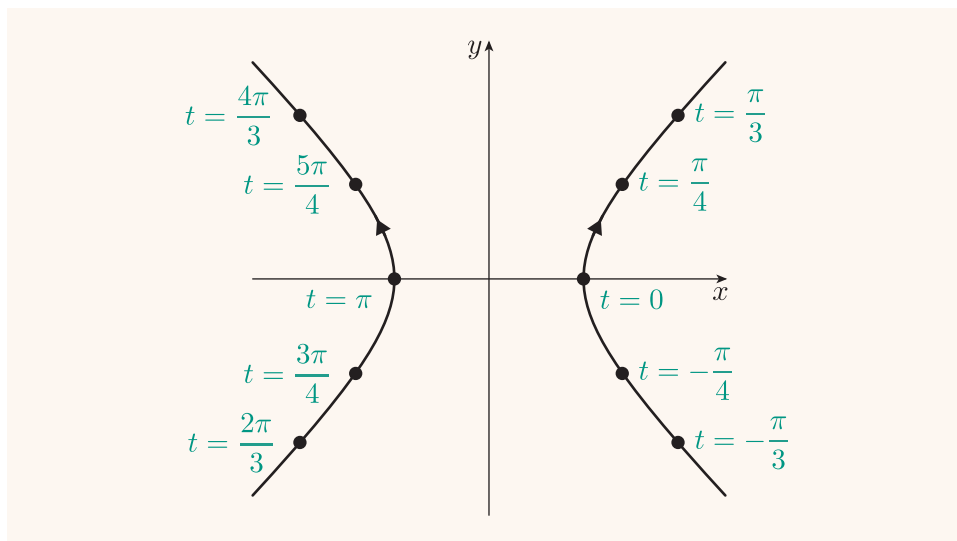


Figure 67 Points on a hyperbola in standard position

So we have the following parametrisation of a hyperbola in standard position.

Standard parametrisation of a hyperbola in standard position

The hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (\text{where } a, b > 0)$$

has the parametrisation

$$x = a \sec t, \quad y = b \tan t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}, \frac{\pi}{2} < t < \frac{3\pi}{2} \right).$$

The values $-\pi/2 < t < \pi/2$ give the right-hand branch, and the values $\pi/2 < t < 3\pi/2$ give the left-hand branch.

As always with parametrisations, there are many alternative parametrisations of a hyperbola in standard position. For example, you can use the parametric equations above, but with a different range of values of t , such as $0 \leq t \leq 2\pi$ excluding $\pi/2$ and $3\pi/2$. As another example, an alternative parametrisation of just the right-hand branch of the hyperbola is given by the equations

$$x = a\sqrt{t^2 + 1}, \quad y = bt.$$

Activity 39 Parametrising a hyperbola in standard position

Write down the standard parametrisation of the hyperbola with equation

$$4x^2 - 64y^2 = 1.$$

Parametric equations for translated curves

If you have a parametrisation of a particular curve, then it's straightforward to find a parametrisation of any curve obtained by translating it. If you translate the original curve by p units right and q units up (where p and q can be positive, negative or zero), then to obtain a parametrisation of the new curve you simply add p to the x -coordinate and q to the y -coordinate in the parametrisation of the original curve.

You saw examples of this when you saw how to obtain parametric equations for circles in Subsection 6.3. As another example, if you translate the ellipse given by the parametrisation

$$x = 3 \cos t, \quad y = 2 \sin t \quad (0 \leq t \leq 2\pi)$$

by 9 units right and 4 units up, then the resulting ellipse has the parametrisation

$$x = 9 + 3 \cos t, \quad y = 4 + 2 \sin t \quad (0 \leq t \leq 2\pi).$$

This is illustrated in Figure 68.

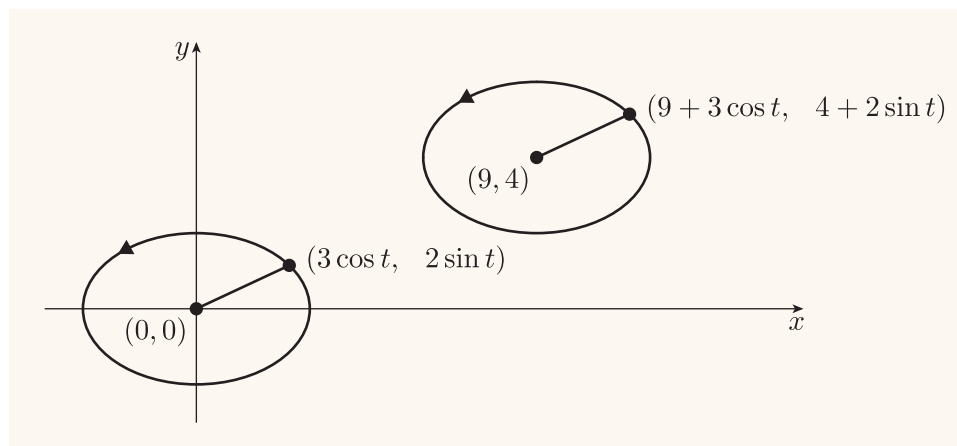


Figure 68 An ellipse and an ellipse obtained by translating the original ellipse

You're asked to find parametrisations of some translated curves in the next activity.

Activity 40 Parametrising translated curves

Find parametrisations of each of the following curves.

- The hyperbola with equation $\frac{x^2}{4} - y^2 = 1$ translated 3 units right and 5 units down.
- The parabola with equation $y^2 = 16x$ translated 1 unit left and 4 units up.

Finding parametrisations of curves obtained from other curves by using transformations other than translations, such as rotations and reflections, is more tricky and beyond the scope of this module. It's best dealt with by using *matrices*.

Using parametric equations for conics on a computer

In the final activity of this unit, you can learn how to use the module computer algebra system to plot conics from parametric equations. You need to be careful when plotting hyperbolas, since, as you've seen, the standard parametric equations $x = a \sec t$, $y = b \tan t$ for a hyperbola are not defined for all values of the parameter t .

Activity 41 Using a computer to plot conics from parametric equations



Work through Subsection 5.2 of the *Computer algebra guide*.

In this unit you've met conic sections and some of their properties. You've also learned about some practical applications of these curves, and some situations where they occur naturally. In the final section you were introduced to parametric equations, a powerful way of representing lines and curves algebraically, which allows you to describe many more curves concisely and conveniently.

Learning outcomes

After studying this unit, you should be able to:

- describe how conics are obtained by slicing a double cone
- use facts about a particular conic in standard position, such as its focus and directrix, to find other facts about it, such as its equation
- sketch ellipses, parabolas and hyperbolas in standard position from their equations
- determine whether a conic with an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is a circle, ellipse, parabola or hyperbola
- find and work with parametrisations of straight lines
- specify the standard parametrisation of a given ellipse, parabola or hyperbola in standard position
- write down parametric equations for a translated curve, given parametric equations for the original curve.

Solutions to activities

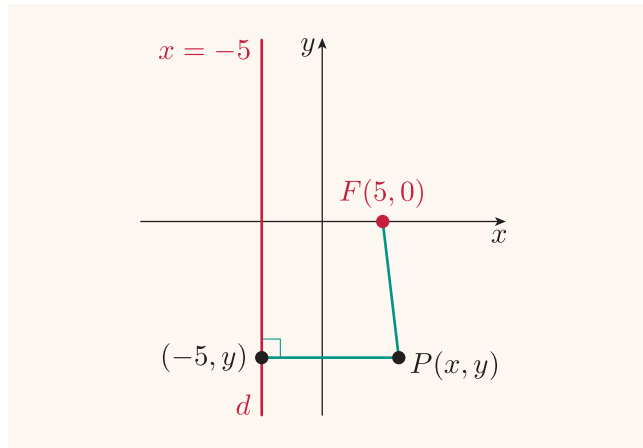
Solution to Activity 1

The solution is in the main text after the activity.

Solution to Activity 3

Let F be the focus, $(5, 0)$, and let d be the directrix, $x = -5$.

Let $P(x, y)$ be any point in the plane.



The distance of P from F is

$$PF = \sqrt{(x - 5)^2 + (y - 0)^2} = \sqrt{(x - 5)^2 + y^2}.$$

The distance of P from d is its distance from the point $(-5, y)$, which is

$$Pd = \sqrt{(x - (-5))^2 + (y - y)^2} = \sqrt{(x + 5)^2}.$$

The point $P(x, y)$ lies on the parabola whenever it satisfies the equation

$$PF = Pd;$$

that is,

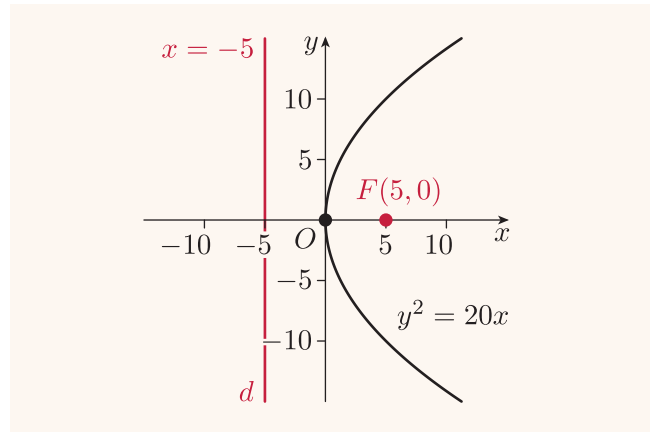
$$\sqrt{(x - 5)^2 + y^2} = \sqrt{(x + 5)^2}.$$

Simplifying gives

$$\begin{aligned} (x - 5)^2 + y^2 &= (x + 5)^2 \\ x^2 - 10x + 25 + y^2 &= x^2 + 10x + 25 \\ y^2 &= 20x. \end{aligned}$$

Hence the equation of the parabola is $y^2 = 20x$.

(This parabola is shown below.)

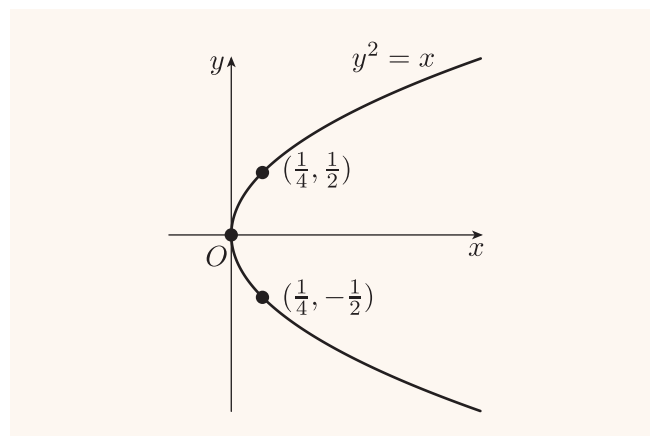


Solution to Activity 4

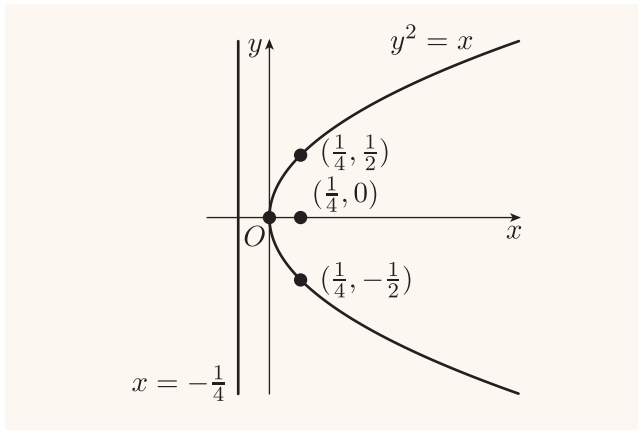
- Here $a = \frac{1}{4}$, so the equation of the parabola is $y^2 = 4 \times \frac{1}{4}x$; that is, $y^2 = x$.
- Here $a = 2$, so the equation of the parabola is $y^2 = 4 \times 2x$; that is, $y^2 = 8x$.

Solution to Activity 5

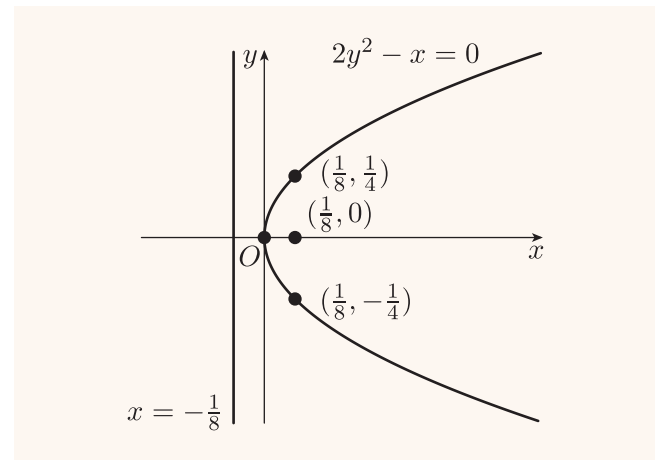
- The equation $y^2 = x$ is of the form $y^2 = 4ax$ with $a = \frac{1}{4}$. Hence it is the equation of a parabola in standard position.
- The vertex is $(0, 0)$. The parabola passes through the points $(a, \pm 2a)$; that is, $(\frac{1}{4}, \pm \frac{1}{2})$.
A sketch of the parabola is shown below.



- (c) The focus is the point $(a, 0)$; that is, $(\frac{1}{4}, 0)$. The directrix is the line $x = -a$; that is, $x = -\frac{1}{4}$. They are shown below.



- (c) The focus is the point $(\frac{1}{8}, 0)$ and the directrix is the line $x = -\frac{1}{8}$, as shown below.



Solution to Activity 6

- (a) The equation is

$$2y^2 - x = 0.$$

Rearranging it to isolate y^2 on the left-hand side gives

$$y^2 = \frac{1}{2}x;$$

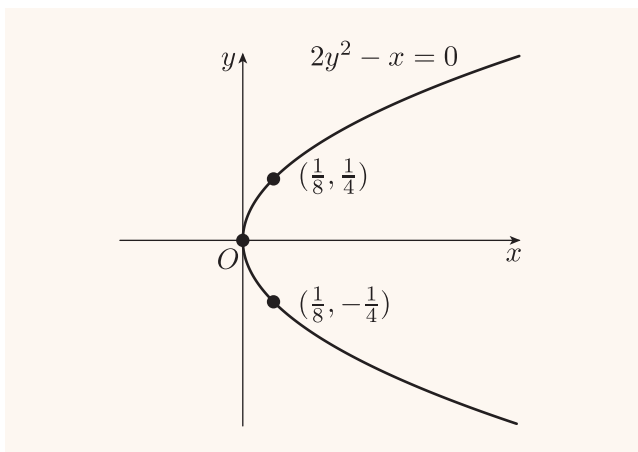
that is,

$$y^2 = 4 \times \frac{1}{8}x.$$

This equation is of the form $y^2 = 4ax$ with $a = \frac{1}{8}$. Hence it is the equation of a parabola in standard position.

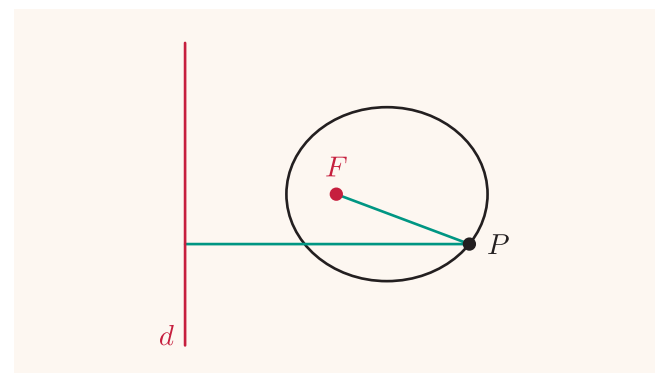
- (b) The vertex is $(0, 0)$. The parabola passes through the points $(\frac{1}{8}, \pm\frac{1}{4})$.

A sketch of the parabola is shown below.



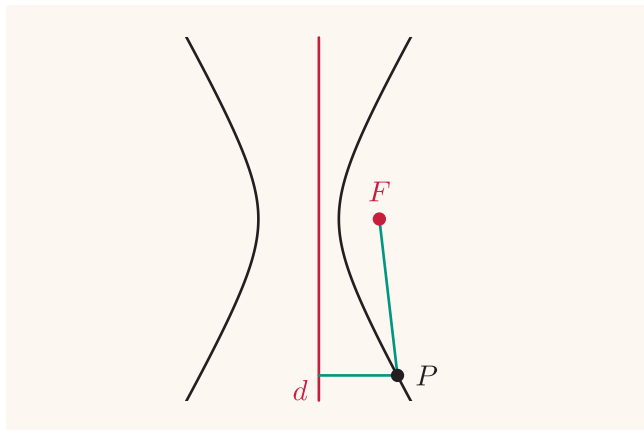
Solution to Activity 7

- (a) (i) When $e = 0$, the 'curve' is a single point, since the only point that satisfies the equation $PF = 0 \cdot Pd$, that is, $PF = 0$, is F itself.
- (ii) When $0 < e < 1$, the curve seems to be an ellipse. An example for $e = 0.5$ is shown below.



- (iii) When $e = 1$, the curve is a parabola, as you know from the previous section.

- (iv) When $e > 1$, the curve has two parts, and seems to be a hyperbola. An example for $e = 2$ is shown below.



Solution to Activity 8

The ellipse in standard position that has eccentricity e and which passes through the points $(\pm a, 0)$ has focus $(ae, 0)$ and directrix $x = a/e$.

Here, the focus is $(3, 0)$ and the directrix is $x = 12$.

So $ae = 3$ and $a/e = 12$.

Multiplying these equations together gives $a^2 = 36$. Since $a > 0$, this gives $a = 6$.

Substituting $a = 6$ into the equation $ae = 3$ gives $6e = 3$ and hence $e = \frac{1}{2}$.

Substituting $a = 6$ and $e = \frac{1}{2}$ into the equation $b^2 = a^2(1 - e^2)$ gives

$$b^2 = 6^2 \left(1 - \left(\frac{1}{2}\right)^2\right) = 36 \times \frac{3}{4} = 27.$$

Since $b > 0$, this gives $b = \sqrt{27}$.

The equation of the ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

so it is

$$\frac{x^2}{36} + \frac{y^2}{27} = 1.$$

Solution to Activity 9

The given equation is of the form $x^2/a^2 + y^2/b^2 = 1$ with $a^2 = 9$ and $b^2 = 7$. Hence the eccentricity of the ellipse is

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{7}{9}} = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}.$$

Solution to Activity 10

- (a) The equation is

$$9x^2 + 16y^2 = 144.$$

Dividing through by 144 to obtain 1 on the right-hand side gives

$$\frac{x^2}{16} + \frac{y^2}{9} = 1;$$

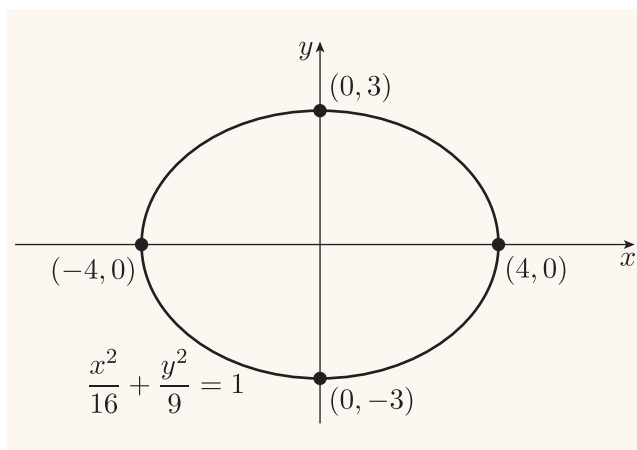
that is,

$$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1.$$

This equation is of the form $x^2/a^2 + y^2/b^2 = 1$, with $a = 4$ and $b = 3$. Hence it is the equation of an ellipse in standard position.

- (b) The vertices are $(\pm 4, 0)$ and $(0, \pm 3)$.

The ellipse is shown below.



- (c) The eccentricity is

$$\begin{aligned} e &= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} \\ &= \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}. \end{aligned}$$

Hence

$$ae = 4 \times \frac{\sqrt{7}}{4} = \sqrt{7},$$

so the foci are $(\pm\sqrt{7}, 0)$; that is, approximately $(\pm 2.6, 0)$.

Also

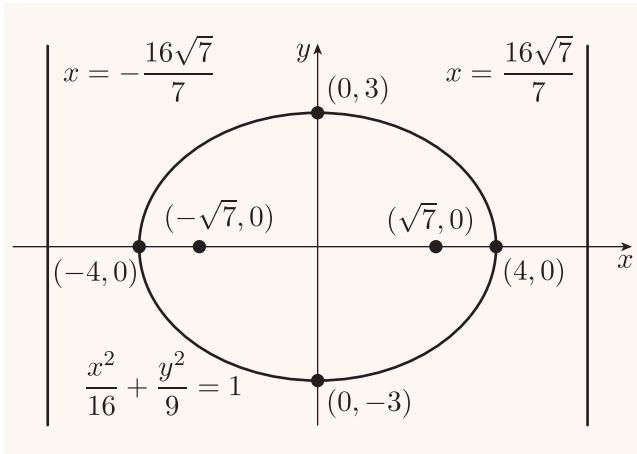
$$\frac{a}{e} = \frac{4}{\sqrt{7}/4} = \frac{16}{\sqrt{7}} = \frac{16\sqrt{7}}{7},$$

so the directrices are the lines

$$x = \pm \frac{16\sqrt{7}}{7};$$

that is, approximately $x = \pm 6.0$.

The ellipse is shown below with its foci and directrices.



Solution to Activity 11

(a) The equation is

$$x^2 + 4y^2 - 4 = 0.$$

Rearranging it to obtain the constant term on the right-hand side gives

$$x^2 + 4y^2 = 4.$$

Dividing through by 4 gives

$$\frac{x^2}{4} + y^2 = 1;$$

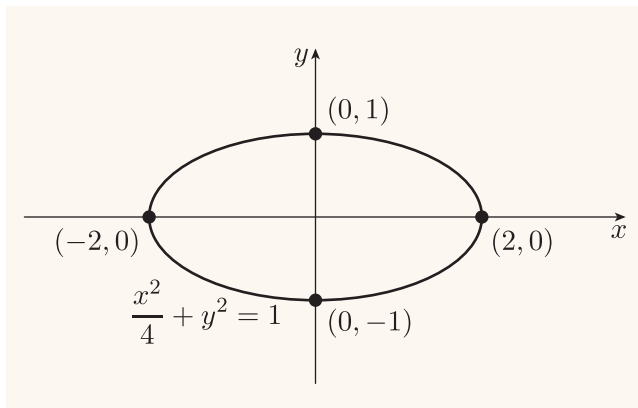
that is,

$$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1.$$

This equation is of the form $x^2/a^2 + y^2/b^2 = 1$, with $a = 2$ and $b = 1$. Hence it is the equation of an ellipse in standard position.

(b) The vertices are $(\pm 2, 0)$ and $(0, \pm 1)$.

The ellipse is shown below.



(c) The eccentricity is

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

Hence

$$ae = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3},$$

so the foci are $(\pm\sqrt{3}, 0)$; that is, approximately $(\pm 1.7, 0)$.

Also

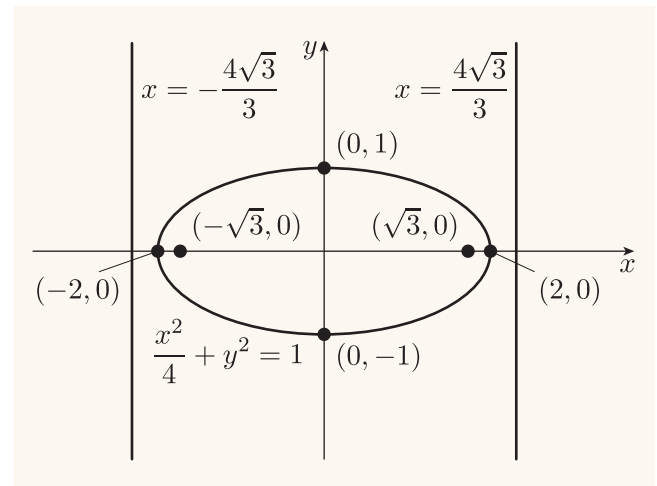
$$\frac{a}{e} = \frac{2}{\sqrt{3}/2} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3},$$

so the directrices are the lines

$$x = \pm \frac{4\sqrt{3}}{3};$$

that is, approximately $x = \pm 2.3$.

The ellipse is shown below with its foci and directrices.



Solution to Activity 12

Dividing the equation $16x^2 + 9y^2 = 144$ through by 144 gives

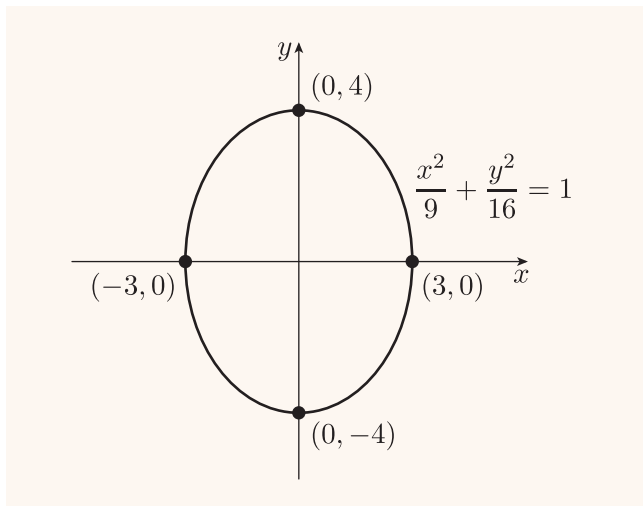
$$\frac{x^2}{9} + \frac{y^2}{16} = 1;$$

that is,

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

The vertices are $(\pm 3, 0)$ and $(0, \pm 4)$.

The ellipse is shown below.



Solution to Activity 13

- (a) By the property mentioned before the activity, a string of length $2a$ gives an ellipse whose major axis has length $2a$. Since an ellipse with a major axis of length 4 m is required, the string must also have length 4 m.
- (b) The distance between the two pegs is the distance between the foci of the ellipse. This is given by $2ae$, where $2a$ is the length of the major axis of the ellipse, which in this case is 4 m, and e is its eccentricity. We can calculate the value of e for the required ellipse by using the formula $e = \sqrt{1 - b^2/a^2}$, where $2b$ is the length of the minor axis of the ellipse, which in this case is 3 metres. So $a = 2$ and $b = \frac{3}{2}$, which gives

$$e = \sqrt{1 - \frac{(\frac{3}{2})^2}{2^2}} = \sqrt{1 - \frac{9}{16}} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}.$$

Hence

$$2ae = 2 \times 2 \times \frac{\sqrt{7}}{4} = \sqrt{7} = 2.65 \text{ (to 2 d.p.)}.$$

So the distance in metres between the pegs must be 2.65 m, to the nearest cm.

Solution to Activity 14

The length of the loop must be the length of the string in Activity 13, plus the distance between the pegs, which by the solution to Activity 13 is

$$(4 + \sqrt{7}) \text{ m} = 6.65 \text{ m (to the nearest cm)}.$$

Solution to Activity 15

- (a) From the diagram,

$$PF = f - a \quad \text{and} \quad Pd = a - g.$$

Since $PF = ePd$, we have

$$f - a = e(a - g). \quad (10)$$

- (b) Similarly,

$$QF = f + a \quad \text{and} \quad Qd = a + g.$$

Since $QF = eQd$, we have

$$f + a = e(a + g). \quad (11)$$

- (c) Adding equations (10) and (11) gives

$$2f = 2ea$$

and hence

$$f = ae.$$

Subtracting equation (10) from equation (11) gives

$$2a = 2eg$$

and hence

$$g = \frac{a}{e}.$$

So the focus F has coordinates $(ae, 0)$ and the directrix d has equation $x = a/e$.

Solution to Activity 16

The hyperbola in standard position that has eccentricity e and passes through the points $(\pm a, 0)$ has focus $(ae, 0)$ and directrix $x = a/e$.

Here, the focus is $(5, 0)$ and the directrix is $x = \frac{9}{5}$. So

$$ae = 5 \quad \text{and} \quad \frac{a}{e} = \frac{9}{5}.$$

Multiplying these equations together gives $a^2 = 9$, so $a = 3$.

Substituting $a = 3$ into the equation $ae = 5$ gives $e = \frac{5}{3}$.

Substituting $a = 3$ and $e = \frac{5}{3}$ into the equation $b^2 = a^2(e^2 - 1)$ gives

$$b^2 = 3^2 \left(\left(\frac{5}{3} \right)^2 - 1 \right) = 5^2 - 3^2 = 16.$$

Since $b > 0$, this gives $b = 4$.

The equation of the hyperbola is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

so it is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1.$$

Solution to Activity 17

The given equation is of the form $x^2/a^2 - y^2/b^2 = 1$ with $a^2 = 9$ and $b^2 = 7$. Hence the eccentricity of the hyperbola is

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{7}{9}} = \sqrt{\frac{16}{9}} = \frac{4}{3}.$$

Solution to Activity 18

(a) The equation is

$$9x^2 - 16y^2 = 144.$$

Dividing through by 144 gives

$$\frac{x^2}{16} - \frac{y^2}{9} = 1;$$

that is,

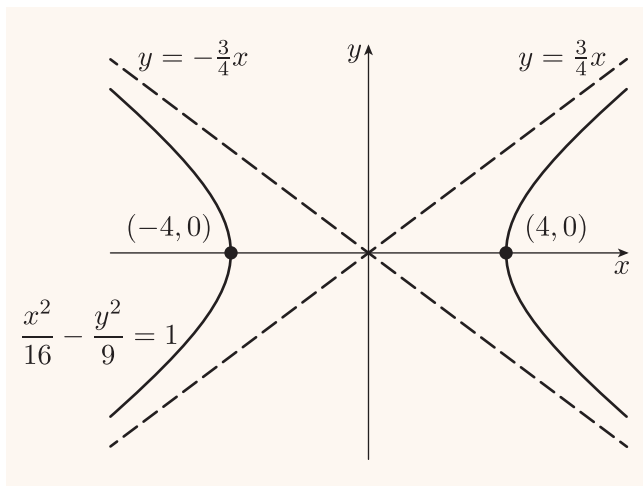
$$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1.$$

This equation is of the form $x^2/a^2 - y^2/b^2 = 1$, with $a = 4$ and $b = 3$. Hence it is the equation of a hyperbola in standard position.

(b) The vertices are $(\pm 4, 0)$.

The asymptotes are the lines $y = \pm \frac{3}{4}x$.

The hyperbola is shown below.



(c) The eccentricity is

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}.$$

Hence

$$ae = 4 \times \frac{5}{4} = 5,$$

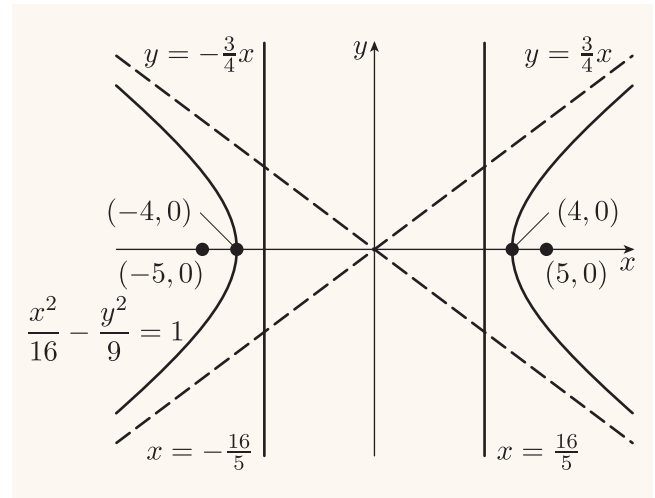
so the foci are $(\pm 5, 0)$.

Also

$$\frac{a}{e} = \frac{4}{5/4} = \frac{16}{5},$$

so the directrices are the lines $x = \pm \frac{16}{5}$; that is, $x = \pm 3.2$.

The foci and directrices are shown below.



Solution to Activity 19

(a) The equation is

$$4x^2 - 3y^2 = 1.$$

The right-hand side is 1 already, so there is no need to divide through by a number.

Rewriting the coefficients on the left-hand side as $4 = 1/(\frac{1}{4})$ and $3 = 1/(\frac{1}{3})$ gives

$$\frac{x^2}{(\frac{1}{4})} - \frac{y^2}{(\frac{1}{3})} = 1;$$

that is,

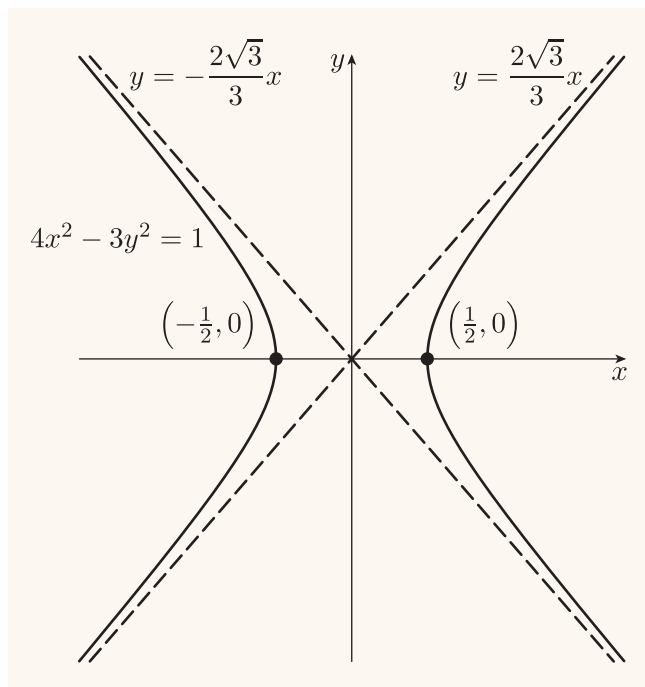
$$\frac{x^2}{(\frac{1}{2})^2} - \frac{y^2}{(1/\sqrt{3})^2} = 1.$$

This equation is of the form $x^2/a^2 - y^2/b^2 = 1$, with $a = \frac{1}{2}$ and $b = 1/\sqrt{3} = \sqrt{3}/3$. Hence it is the equation of a hyperbola in standard position.

(b) The vertices are $(\pm\frac{1}{2}, 0)$.

The asymptotes are the lines $y = \pm\frac{2\sqrt{3}}{3}x$; that is, approximately $y = \pm 1.2x$.

The hyperbola is shown below.



(c) The eccentricity is

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1/3}{1/4}} = \sqrt{1 + \frac{4}{3}} = \sqrt{\frac{7}{3}} = \frac{\sqrt{21}}{3}.$$

Hence

$$ae = \frac{1}{2} \times \frac{\sqrt{21}}{3} = \frac{\sqrt{21}}{6},$$

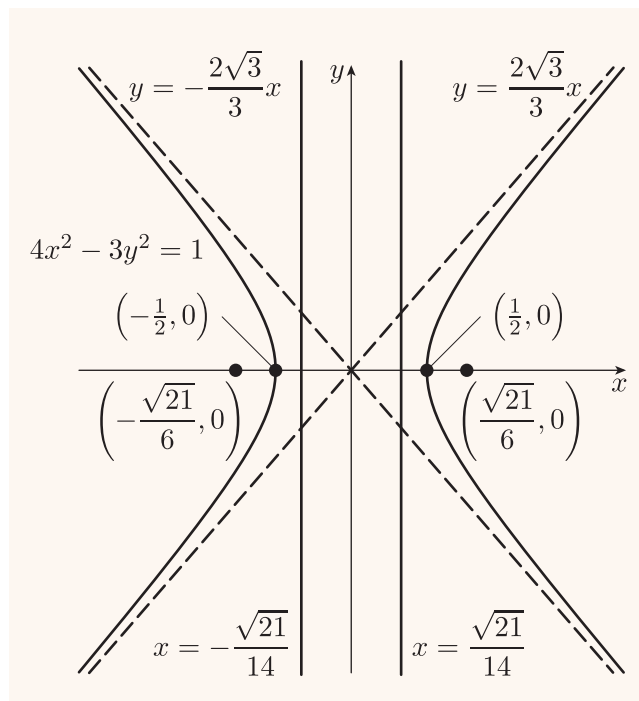
so the foci are $(\pm\frac{\sqrt{21}}{6}, 0)$; that is, approximately $(\pm 0.76, 0)$.

Also

$$\frac{a}{e} = \frac{1}{2} \div \frac{\sqrt{21}}{3} = \frac{1}{2} \times \frac{3}{\sqrt{21}} = \frac{3\sqrt{21}}{2 \times 21} = \frac{\sqrt{21}}{14},$$

so the directrices are the lines $x = \pm\frac{\sqrt{21}}{14}$; that is, approximately $x = \pm 0.33$.

The foci and directrices are shown below.



Solution to Activity 20

(a) The equation is

$$4y^2 - 25x^2 + 1 = 0.$$

Rearranging it so that 1 appears on the right-hand side gives

$$25x^2 - 4y^2 = 1.$$

Rewriting the coefficients on the left-hand side as $25 = 1/(\frac{1}{25})$ and $4 = 1/(\frac{1}{4})$ gives

$$\frac{x^2}{(\frac{1}{25})} - \frac{y^2}{(\frac{1}{4})} = 1;$$

that is,

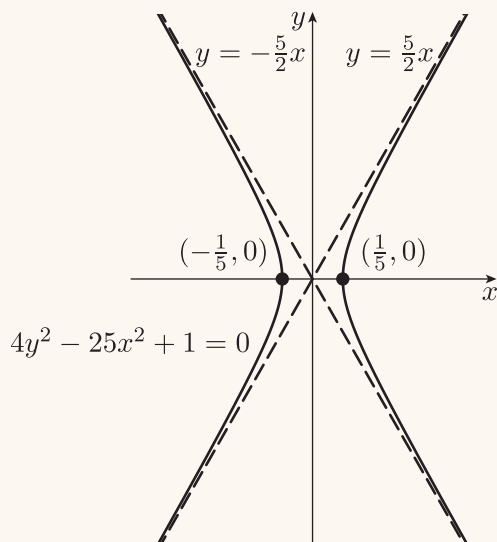
$$\frac{x^2}{(\frac{1}{5})^2} - \frac{y^2}{(\frac{1}{2})^2} = 1.$$

This equation is of the form $x^2/a^2 - y^2/b^2 = 1$, with $a = \frac{1}{5}$ and $b = \frac{1}{2}$. So it represents a hyperbola.

The vertices are $(\pm\frac{1}{5}, 0)$.

The asymptotes are the lines $y = \pm\frac{5}{2}x$.

The hyperbola is as follows.



(b) The equation is

$$3x - y^2 = 0.$$

It can be rearranged as

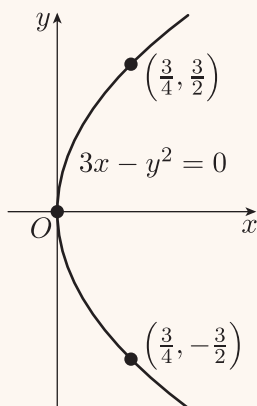
$$y^2 = 3x$$

and then as

$$y^2 = (4 \times \frac{3}{4})x.$$

This equation is of the form $y^2 = 4ax$ with $a = \frac{3}{4}$, so it represents a parabola in standard position.

The parabola passes through the points $(0,0)$ and $(\frac{3}{4}, \pm\frac{3}{2})$. It is shown below.



(c) The equation is

$$x^2 + 18y^2 - 9 = 0.$$

Rearranging it so that the constant term appears on the right-hand side gives

$$x^2 + 18y^2 = 9.$$

Dividing through by 9 gives

$$\frac{x^2}{9} + 2y^2 = 1;$$

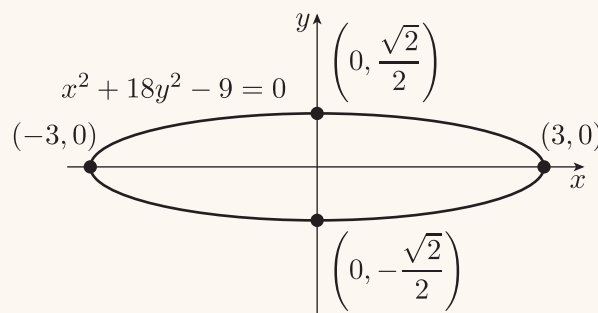
that is,

$$\frac{x^2}{3^2} + \frac{y^2}{(1/\sqrt{2})^2} = 1.$$

This equation is of the form $x^2/a^2 + y^2/b^2 = 1$, with $a = 3$ and $b = 1/\sqrt{2} = \sqrt{2}/2$. Therefore it represents an ellipse.

The vertices of the ellipse are $(\pm 3, 0)$ and $(0, \pm\sqrt{2}/2)$. The second pair of vertices are approximately $(0, \pm 0.71)$.

The ellipse is shown below.



Solution to Activity 21

The given equation

$$3x^2 + 7xy + 4y^2 - 48x - 48y + 324 = 0$$

is of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

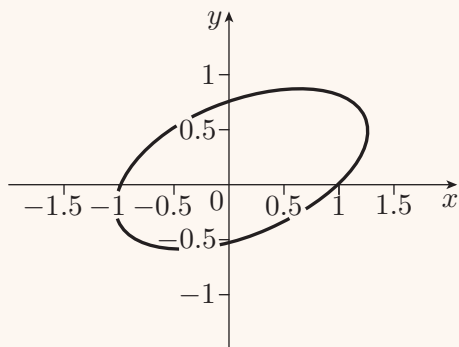
with $A = 3$, $B = 7$ and $C = 4$. This gives

$$B^2 - 4AC = 7^2 - 4 \times 3 \times 4 = 49 - 48 = 1.$$

Since $B^2 - 4AC > 0$, the conic is a hyperbola.

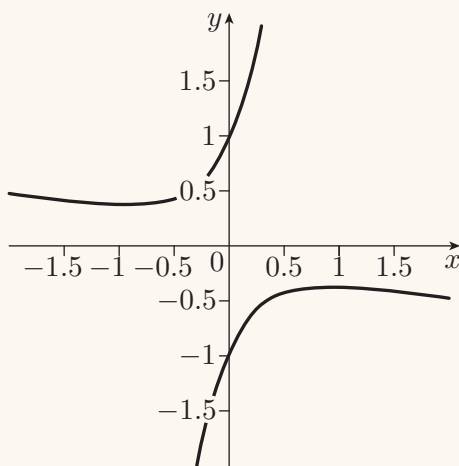
Solution to Activity 22

- (a) The graph of the equation
 $2x^2 - 3xy + 5y^2 - y - 2 = 0$
 is shown below.



The equation is of the form
 $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ with
 $A = 2$, $B = -3$ and $C = 5$. This gives
 $B^2 - 4AC = (-3)^2 - 4 \times 2 \times 5 = 9 - 40 = -31$.
 Since $B^2 - 4AC < 0$, and $A \neq C$, the conic is
 an ellipse.

- (b) The graph of the equation
 $x^2 + 5xy - y^2 + 1 = 0$
 is shown below.



The equation is of the form
 $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ with
 $A = 1$, $B = 5$ and $C = -1$. This gives
 $B^2 - 4AC = 5^2 - 4 \times 1 \times (-1) = 25 + 4 = 29$.
 Since $B^2 - 4AC > 0$, the conic is a hyperbola.

(Details of how to use the CAS to plot these conics are given in the *Computer algebra guide*, in the section 'Computer methods for CAS activities in Books A–D'.)

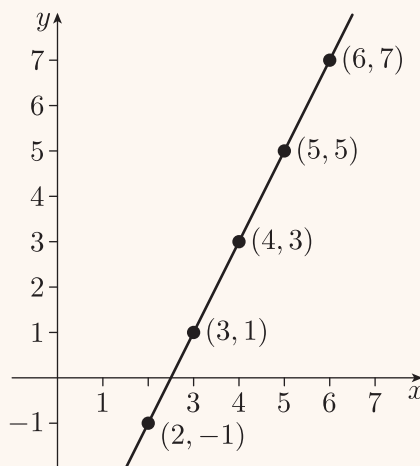
Solution to Activity 23

- (a) The table below shows the values of x and y for the given values of t .

t	-1	0	1	2
x	3	4	5	6
y	1	3	5	7

So the required points are $(3, 1)$, $(4, 3)$, $(5, 5)$ and $(6, 7)$.

- (b) The graph is shown below. The points appear to lie on a straight line.



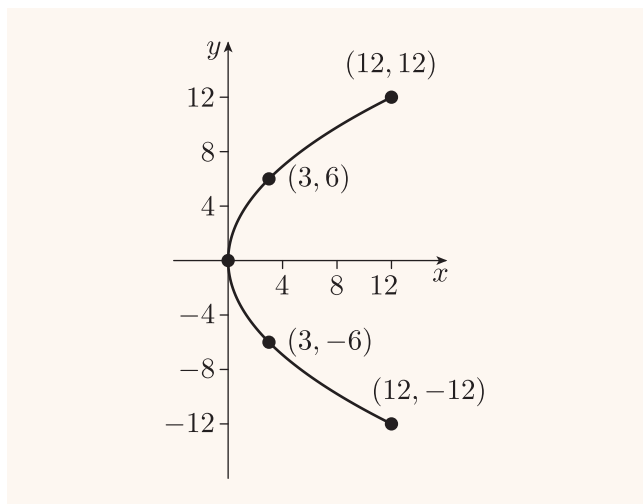
Solution to Activity 24

- (a) The table below shows the values of x and y for the given values of t .

t	-2	-1	0	1	2
x	12	3	0	3	12
y	-12	-6	0	6	12

So the required points are $(12, -12)$, $(3, -6)$, $(0, 0)$, $(3, 6)$ and $(12, 12)$.

- (b) Plotting the points from part (a) gives the following sketch.



(This curve stops at the endpoints (12, -12) and (12, 12), because of the restricted range of values of the parameter t .)

Solution to Activity 26

- (a) You will need to take t to be at least 18.8 to trace out the whole curve.
- (b) The curve appears to be an ellipse. (Activity 38 later in the unit will help you see why an ellipse is plotted.)
- (c) The following values give the second two hypotrochoids in Figure 51.

For the second hypotrochoid:

$$R = 1.3, r = 0.9, d = 1 \text{ and } t = 57.$$

For the third hypotrochoid:

$$R = 1.9, r = 1, d = 0.25 \text{ and } t = 63.$$

Solution to Activity 27

The equation $x = 3t + 1$ gives $t = \frac{1}{3}(x - 1)$. Hence

$$y = -12 \times \frac{1}{3}(x - 1) + 5 = -4(x - 1) + 5 = -4x + 9.$$

So the parametric equations give the line with equation $y = -4x + 9$.

Solution to Activity 28

The given parametric equations $x = 1 - 2t$, $y = t + 1$ are of the form $x = at + b$, $y = ct + d$, so with the given range of values for t they describe part of a straight line.

The values of t are given by $-1 < t \leq 3$. Taking $t = -1$ gives

$$x = 1 - 2(-1) = 3 \quad \text{and} \quad y = -1 + 1 = 0,$$

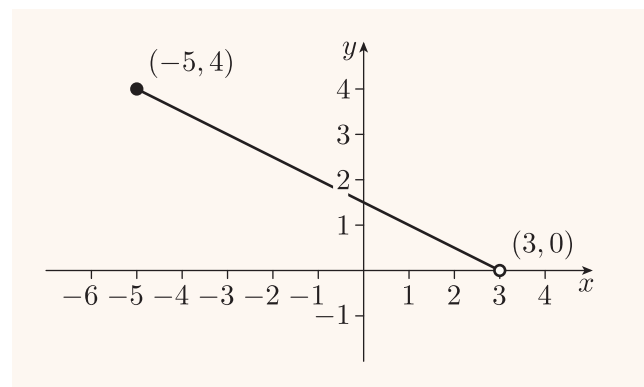
so an endpoint of the graph is (3, 0). This endpoint is excluded.

Taking $t = 3$ gives

$$x = 1 - 2 \times 3 = -5 \quad \text{and} \quad y = 3 + 1 = 4,$$

so the other endpoint is (-5, 4). This endpoint is included.

The graph is shown here.



Solution to Activity 29

- (a) (iii) The value of t that corresponds to the midpoint is the value halfway between the endpoints -2 and 0 of the range of values of t ; that is, it is $t = -1$.
- (b) (iii) The value of t that corresponds to the midpoint is the value halfway between the endpoints -1 and 3 of the range of values of t ; that is, it is $t = 1$.

The value of t that corresponds to the point that's a quarter of the way along the line segment from the endpoint corresponding to $t = -1$ to the endpoint corresponding to $t = 3$ is the value that's a quarter of the way from the endpoint -1 to the endpoint 3 of the range of values of t ; that is, it is $t = 0$.

(These results are explained in the text after the activity.)

Solution to Activity 30

- (a) Although the paths of the ships cross each other, the ships don't collide, as they reach the crossing point at different times.
- (b) The closest distance is about 2.24 km, which occurs after about 5 hours.

Solution to Activity 31

At time t , the distance d between the two aircraft is given by

$$\begin{aligned} d^2 &= (500t + 10 - (496t + 22))^2 \\ &\quad + (-312t - 12 - (-310t - 23))^2 \\ &= (4t - 12)^2 + (-2t + 11)^2 \\ &= 16t^2 - 96t + 144 + 4t^2 - 44t + 121 \\ &= 20t^2 - 140t + 265. \end{aligned}$$

Completing the square gives

$$\begin{aligned} d^2 &= 20(t^2 - 7t) + 265 \\ &= 20\left(\left(t - \frac{7}{2}\right)^2 - \frac{49}{4}\right) + 265 \\ &= 20\left(t - \frac{7}{2}\right)^2 + 20. \end{aligned}$$

The term $20\left(t - \frac{7}{2}\right)^2$ never takes a value less than 0, and it takes the value 0 when $t = \frac{7}{2}$. At this value of t , the value of d^2 is 20. Hence the minimum value of d is $\sqrt{20} = 4.47$ (to 2 d.p.).

So the closest distance is approximately 4.47 km, and it occurs 3.5 hours after the start time.

Solution to Activity 32

By the box shortly before the activity, suitable parametric equations are

$$x = -2 + t, \quad y = 3 + 4t.$$

(There are other parametrisations of this line.)

Solution to Activity 33

- (a) By the box before the activity, the line has parametric equations

$$x = x_0 + t(x_1 - x_0), \quad y = y_0 + t(y_1 - y_0)$$

where $(x_0, y_0) = (1, 5)$ and $(x_1, y_1) = (4, -7)$.

These parametric equations are

$$x = 1 + t(4 - 1), \quad y = 5 + t(-7 - 5);$$

that is,

$$x = 1 + 3t, \quad y = 5 - 12t.$$

(If you take the points in the other order, then you obtain the parametric equations

$$x = 4 - 3t, \quad y = -7 + 12t.$$

There are other parametrisations too.)

- (b) By the box before the activity, the line segment joining the points (x_0, y_0) and (x_1, y_1) has parametric equations

$$x = x_0 + t(x_1 - x_0), \quad y = y_0 + t(y_1 - y_0)$$

with t restricted to $0 \leq t \leq 1$.

Here $(x_0, y_0) = (-2, 4)$ and $(x_1, y_1) = (3, 1)$, so the line segment has the parametrisation

$$\begin{aligned} x &= -2 + t(3 - (-2)), \quad y = 4 + t(1 - 4) \\ &\quad (0 \leq t \leq 1); \end{aligned}$$

that is,

$$x = -2 + 5t, \quad y = 4 - 3t \quad (0 \leq t \leq 1).$$

(If you take the points in the other order, then you obtain the parametrisation

$$x = 3 - 5t, \quad y = 1 + 3t \quad (0 \leq t \leq 1).$$

There are other parametrisations too.)

Solution to Activity 34

- (a) A suitable parametrisation is

$$x = -3 + 4 \cos t, \quad y = 1 + 4 \sin t \quad (0 \leq t \leq 2\pi).$$

(There are many other possible parametrisations. Also, the restriction on the values of t is not necessary, but is included here to avoid repeating points other than the point corresponding to $t = 0$.)

- (b) The semicircle has centre $(2, 1)$ and radius 2.

A suitable pair of parametric equations are

$$x = 2 + 2 \cos t, \quad y = 1 + 2 \sin t.$$

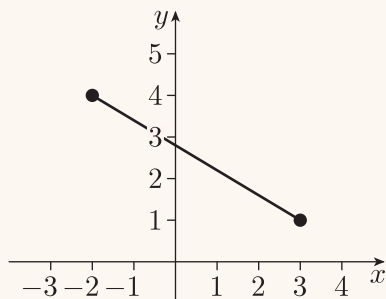
For these parametric equations to describe the semicircle, rather than a whole circle, we must restrict the values of t . A suitable parametrisation is

$$x = 2 + 2 \cos t, \quad y = 1 + 2 \sin t \quad \left(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\right).$$

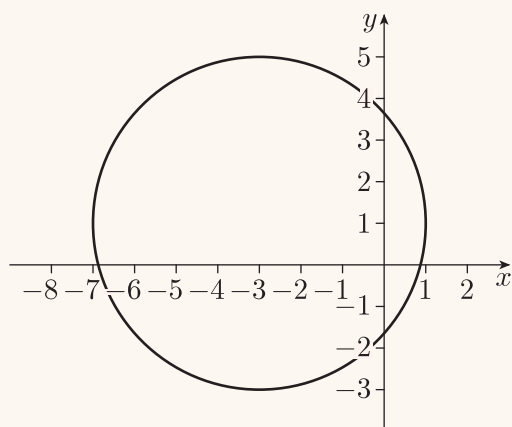
(There are other ranges of values of t that will do just as well, such as $\frac{3\pi}{2} \leq t \leq \frac{5\pi}{2}$.)

Solution to Activity 35

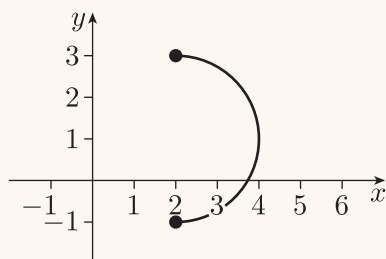
(a)



(b)



(c)



(Details of how to use the CAS to plot these lines and curves are given in the *Computer algebra guide*, in the section ‘Computer methods for CAS activities in Books A–D’.)

Solution to Activity 36

(a) Since $a = \frac{1}{2}$, the standard parametrisation is

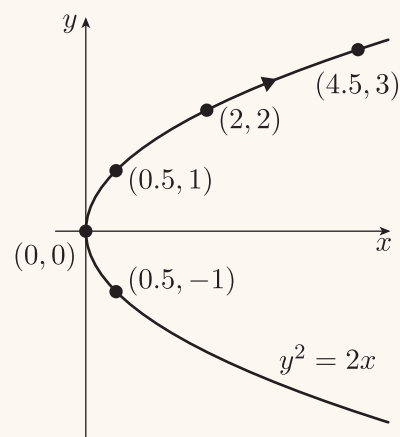
$$x = \frac{1}{2}t^2, \quad y = t.$$

(b) The values of x and y for the given values of t are shown in the following table.

t	-1	0	1	2	3
x	0.5	0	0.5	2	4.5
y	-1	0	1	2	3

Thus the required points are $(0.5, -1)$, $(0, 0)$, $(0.5, 1)$, $(2, 2)$ and $(4.5, 3)$.

(c) The parabola is as shown below.



Solution to Activity 37

Here $a = 3$ and $b = 2$, so the standard parametrisation is

$$x = 3 \cos t, \quad y = 2 \sin t \quad (0 \leq t \leq 2\pi).$$

Solution to Activity 38

- (a) When $R = 1$, $r = 0.5$ and $d = 1$, the parametric equations for the hypotrochoid are

$$\begin{aligned}x &= (1 - 0.5) \cos t + \cos\left(\frac{1 - 0.5}{0.5}t\right) \\&= 0.5 \cos t + \cos t \\&= 1.5 \cos t\end{aligned}$$

and

$$\begin{aligned}y &= (1 - 0.5) \sin t - \sin\left(\frac{1 - 0.5}{0.5}t\right) \\&= 0.5 \sin t - \sin t \\&= -0.5 \sin t,\end{aligned}$$

as required.

- (b) The equations found in part (a) are nearly of the form of the parametric equations in the standard parametrisation of an ellipse in standard position, except that the coefficient of $\sin t$ in the equation for y is negative.

Changing this coefficient to its negative gives the parametric equations

$$x = 1.5 \cos t, \quad y = 0.5 \sin t, \quad (12)$$

which give an ellipse in standard position.

Since the x -axis is an axis of symmetry of this ellipse, changing the coefficient 0.5 in the equation for y back to -0.5 doesn't change the shape of the curve given by the equations. So the equations found in part (a) give the same ellipse as equations (12).

(The two pairs of parametric equations trace out the ellipse in opposite directions.)

Solution to Activity 39

The equation $4x^2 - 64y^2 = 1$ can be written as

$$\frac{x^2}{\left(\frac{1}{4}\right)} - \frac{y^2}{\left(\frac{1}{64}\right)} = 1.$$

So it is of the form $x^2/a^2 - y^2/b^2 = 1$ with $a = \frac{1}{2}$ and $b = \frac{1}{8}$. Hence the standard parametrisation is

$$x = \frac{1}{2} \sec t, \quad y = \frac{1}{8} \tan t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}, \frac{\pi}{2} < t < \frac{3\pi}{2}\right).$$

Solution to Activity 40

- (a) The equation $x^2/4 - y^2 = 1$ represents the hyperbola in standard position with $a = 2$ and $b = 1$. It has the standard parametrisation

$$\begin{aligned}x &= 2 \sec t, \quad y = \tan t \\&\quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}, \frac{\pi}{2} < t < \frac{3\pi}{2}\right).\end{aligned}$$

Translating this hyperbola 3 units right and 5 units down gives the hyperbola with parametrisation

$$\begin{aligned}x &= 3 + 2 \sec t, \quad y = -5 + \tan t \\&\quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}, \frac{\pi}{2} < t < \frac{3\pi}{2}\right).\end{aligned}$$

- (b) The equation $y^2 = 16x$ represents the parabola in standard position with $a = 4$. It has the standard parametrisation

$$x = 4t^2, \quad y = 8t.$$

Translating this parabola 1 unit left and 4 units up gives the parabola with parametrisation

$$x = -1 + 4t^2, \quad y = 4 + 8t.$$

Acknowledgements

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